REPLICATING PORTFOLIOS: CALIBRATION TECHNIQUES FOR THE CALCULATION OF THE SOLVENCY II ECONOMIC CAPITAL

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Abstract:

When endeavoring to build an internal model, life insurance companies are often faced with the choice of which method to use for the distribution of shareholder's equity over one year. However, the highly stochastic nature of this type of approach can sometimes lead to significant calculation times that threaten to compromise its operational implementing.

The use of calculation acceleration or approximation techniques therefore appears as essential to the application of such methods. One possible approach is the use of the Replicating Portfolios technique, in which the projection times are strongly reduced by an estimation of shareholder's equity based on a portfolio of assets that replicates the best estimate of the company's liabilities.

However, the calibration of the Replicating Portfolios method presents several difficulties that may lead to unsatisfactory results. In this article, we introduce a calibration technique that was developed in order to guarantee the robustness of the estimation of the Solvency II economic capital.

Keywords: Replicating Portfolios, economic capital, Solvency II, parametric form, Nested Simulations, shareholder's equity, risk factors, internal model

1. INTRODUCTION

In the Solvency II context, the economic capital is defined as the shareholder's equity that the company must own in order to withstand economic bankruptcy over one year with a 99.5% threshold. This criterion therefore relies, among others, on the notion of equity distribution over one year. However, in most cases related to life insurance portfolios, the calculation of this distribution remains a delicate matter as taking into

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account the interactions between assets and liabilities is time consuming.

It therefore seems crucial to accelerate the usual methods in order to build an internal model that enables operational calculations of capital requirements. Among the various methods that may be proposed, the Nested Simulations (NS) approach is the best suited to the Solvency II context. However, its use does imply considerable calculation times. Devineau and Loisel (2009) propose an acceleration technique that can be applied to this method in order to produce the same result as the NS approach with a significant reduction of the number of simulations.

In many cases, approximation techniques offer a powerful alternative to methods based on the reduction of the number of simulations. The underlying principle is the construction of a proxy that enables very quick assessments of the items of the balance sheet without having to resort to Monte-Carlo techniques. Methods such as "Parametric form" and "Replicating portfolios" are known examples of capital estimation methods based on approximations. In addition, the Replicating Portfolios approach examined herein consists in the construction of a portfolio of financial assets whose value provides a quick estimation of that of relevant variables (i.e. the shareholder's equity or the best estimate of liabilities). The use of a Replicating Portfolio makes it easy to calculate the distribution of equity at the end of the first period through a simulation of the portfolio's value. The estimation of the market price of the Replicating Portfolio at t=1 is indeed considerably faster than the calculation of equity by means of Monte-Carlo simulations with an ALM model.

Although the use of a Replicating Portfolio may appear to be an efficient way to calculate the Solvency II economic capital, its calibration requires considerable attention. It gives rise to several issues, the first of which being the variable that constitutes the wisest choice: is it the best estimate of liabilities or the value of shareholder's equity? How does one select and set the assets that are to be included in the Replicating Portfolio? Should this matter be decided by an expert or are there automation and optimization methods for the determination of the portfolio? These are the questions that we shall address in this article.

In the first section we shall discuss the issues surrounding the calculation of economic capital in the Solvency II environment. We shall then examine the usual applications of Replicating Portfolios, before describing the alternative approach that we have developed. Finally we shall present an analysis of the different methods, and based on the results of the internal models, we shall highlight the points worthy of focus.

2. THE SOLVENCY II ECONOMIC CAPITAL

2.1 General information

The solvency II economic capital is the total amount of equity that the company must own in order to overcome bankruptcy with a one year timeline and a confidence level of 99.5%. This definition of bankruptcy relies on the study of the weakening of the insurer's balance sheet over a one year period.

Eq_t, is the company's equity at date t. This variable is given by the following subtraction:

$$\mathrm{Eq_t} = \mathrm{A_t} - \mathrm{L_t},$$

With A_t (resp. L_t) being the market value of the asset (resp. the best estimate of liabilities) at t.

For the purpose of clarity, in this article we will not take the Risk Margin into account. Thus, we will not distinguish the "fair value" and "best estimate" of the liabilities.

Under certain assumptions, the economic capital can be calculated on the basis of the following relation:

$$C = Eq_0 - P(0,1).\, q_{0,5\%}(Eq_1),$$

Where P(0,1) is the price at date 0 of a zero-coupon bond with a one year maturity. For the purpose of this article, one will assume the conditions for the validity of this formula as being fulfilled.

The reader should refer to the article by Devineau and Loisel (2009) for a detailed description of the elements introduced above.

2.2 Calculation methods

There are two main categories of methods used to calculate the Solvency II economic capital: the "standard formula" modular methods and the approaches based on the distribution of shareholder’s equity over a one year period (we use indifferently the terms of “shareholder’s equity” or “equity” thereafter).

2.2.1 The "standard formula" approaches

In the context of the "standard formula" method, the economic capital is calculated for each "main risk driver" (stock, rates, mortality). These capitals are then aggregated by means of correlation matrices. This approach may lead to several aggregation levels. For example, the QIS relies on intra-modular and inter-modular aggregation view (see CEIOPS QIS 5 Technical Specifications 2010). In this type of bottom-up approach, elementary capitals are estimated as the difference between central equity and shocked equity. More often than not, these valuations require an ALM model.

2.2.2 *The techniques for the construction of the equity distribution over a one-year period*

The aim of these techniques is to achieve an empirical distribution of the company's equity so as to deduce its economic capital from the following relation:

$$\mathrm{C} = \mathrm{Eq}_0 - \mathrm{P}(0{,}1).\,\mathrm{q}_{0{,}5\%}(\mathrm{Eq}_1)$$

Generally speaking, the calculation of the Eq_0 quantity does not give rise to any operational difficulties (similar to MCEV calculations for life insurance portfolios). However, the estimation of the 0.5% percentile of Eq_1 requires the knowledge of the distribution tail.

In most cases of life-insurance portfolios, the difficulties pertaining to the calculation are raised by the determination of the amount of equity at the end of the first period. Indeed, non-linearities associated with the optional nature of life-insurance liabilities require the use of Monte-Carlo techniques in order to obtain the economic balance sheet after one year.

There are several manners in which these methods may be implemented: the "Nested Simulations" (NS) method, the Replicating Portfolios (RP) method, and the approaches based on "parametric forms".

In the following sections, we offer a description of the NS and RP methods.

2.3 Calculation of the economic capital with the NS approach

The NS method consists in carrying out simulations for the first period before launching for each one of them a new set of simulations in order to determine the distribution of the company's equity at $t = 1$.

It should be noted that the first year simulations (called primary simulations) are real-world simulations, whereas the subsequent simulations (called secondary simulations) are risk-neutral, conditionally to the first period information.

The process can be illustrated with the following diagram:

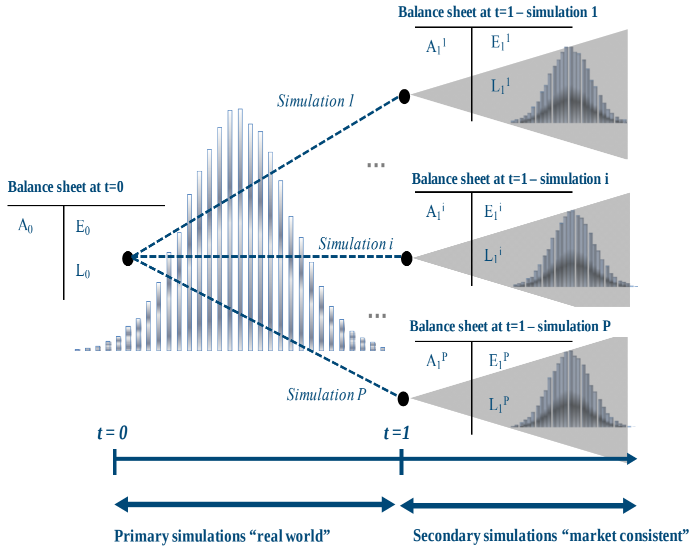

Figure 1 : Equity distribution calculation in the NS methodology

We shall now introduce some notations in order to formalise the calculations carried out in a NS approach:

- $R_u^{p,s}$ the random variable of profit at date $u > 1$ for the primary simulation $p \in \{1, \dots, P\}$ and for the secondary simulation $s \in \{1, \dots, S\}$,
- R_1^p the first period profit for the primary simulation p,
- $\delta_u^{p,s}$ the discount factor at the date $u > 1$ for the primary simulation p and the secondary simulation s,
- δ_1^p the discount factor of the first period for the primary simulation p,
- F_1^p the information of the first year contained within the primary simulation p,
- FP_1^p the shareholder's equity at the end of the first period for the primary simulation p,
- VEP_1^p the best estimate of the liabilities at the end of first period for the primary simulation p,
- A_1^p the market value of the asset at the end of the first period for the primary simulation p.

The equity at $t = 1$, for the primary simulation p , satisfies the following equation:

$$Eq_1^p = R_1^p + E\left[\sum_{u\geq 2} \frac{\delta_u}{\delta_1} R_u \,|F_1^p\right].$$

We shall consider the following estimator of Eq_1^p:

$$\widehat{Eq}_1^p = R_1^p + \frac{1}{S}\sum_{s=1}^{S}\sum_{u\geq 2} \frac{\delta_u^{p,s}}{\delta_1^p} R_u^{p,s}.$$

The determination of $q_{0,5\%}(Eq_1)$ is based generally on a rank-estimation like $Eq_1^{([0,5\% \times P])}$. The NS method is very time-consuming because of its complexity in $P \times S$. However, there are various techniques that enable to efficiently speed up the process. For a detailed description of the NS methodology and of acceleration techniques, the reader should refer to Devineau and Loisel (2009).

An alternative method consists in calculating, using a proxy, the value of the equity or liabilities without resorting to Monte-Carlo simulations for the valuation of these balance sheet items at $t = 1$.

2.4 The Replicating Portfolios method

2.4.1 General information and definitions

The financial literature is rife with examples of replication techniques used to address assets valuation issues. In this section, we provide an overview of the basic principle and we invite the reader to refer to the works of Dana and Jeanblanc-Picqué (1998) for an in-depth presentation of valuing methods based on the construction of financing strategies.

We will take $(F_t)_{t\geq 0}$ as the filtration that characterises the financial information available at time *t* and we shall consider *Z* as a F_T–measurable variable. Let us assume that the economy comprises $d + 1$ financial assets with a price of $X_t = (X_t^0, X_t^1, \dots, X_t^d)$ at date t where X_t^0 represents the price of an asset with no dynamic risk:

$$dX_t^0 = X_t^0 r_t dt, \qquad X_t^0 = 1,$$

With r_t being the risk-free rate.

Given the right assumptions, it is possible to develop an adapted process $w_t = (w_t^0, w_t^1, \dots, w_t^d)$ that enables to replicate the random variable Z, i.e. that satisfies the two properties given here:

$$\begin{cases} w_t \cdot X_t = w_0 \cdot X_0 + \int_0^t w_s \cdot dX_s & \forall t \le T, a.s \\ w_T \cdot X_T = Z & a.s \end{cases}$$

The marginal distributions of the process represent the weight of each asset of a self-financing portfolio that replicates a.s the cash flows of Z.

The price π_t of Z is then equal to: $\pi_t = w_t \cdot X_t = E_Q\left(e^{-\int_t^T r_s ds} \cdot Z | F_t\right)$.

The Black & Scholes formula that enables to determine the call price with a time to maturity of T, a strike price of K and an underlying X_t^1 can be drawn from the valuation of a portfolio that replicates the variable $Z = (X_T^1 - K)^+$.

Let us suppose a risk-free interest rate r_t that is constant and equal to r and that under a risk-neutral probability Q, one has:

$$\frac{dX_t^1}{X_t^1} = rdt + \sigma dW_t,$$

Where $(W_t)_t$ is a Brownian motion under Q. Then the replicating portfolio at date t is as follows:

$$w_t^0 = -K \cdot e^{-rT} \cdot N(d_2^t),$$

And,

$$w_t^1 = N(d_1^t),$$

Where,

$$d_1^t = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{X_t^1}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) \cdot (T-t)\right), \quad d_2^t = d_1^t - \sigma\sqrt{T-t}.$$

The use of these tools for the purpose of life-insurance portfolio valuations is a far more delicate matter. Many authors have developed methodologies to price Financial Options and Guarantees (FOG) embedded in life-insurance products; for further information on the subject the reader should refer to Bacinello (2001) and Bacinello (2005). However, for most life-insurance portfolios, the replication of economic variables calculated by means of an internal model is not operationally achievable using the aforementioned techniques because of specific management regulations and accounting mechanisms that often rule out the use of closed formulae.

The notion of Replicating Portfolios used to address insurance issues has therefore a slightly different definition than that mentioned above. Although the objective is to build a portfolio whose cash flows (or value) are close to the items projected by the internal model (liabilities and profits), the weighting of candidate assets is deterministic and non stochastic.

We have seen above that the weights of a self-financing portfolio constitute an adapted process (these are therefore random variables); this leads to an adjustment at each date t of the quantity of assets owned in relation to the economic conditions at time t. However the use of Replicating Portfolios in the context of insurance systematically relies on a constant asset-mix.

For example, for the purpose of replicating cash-flows, where $cf_{liab}(t)$ (resp. $cf_{RP}^{k}(t)$ with $= 1, \dots, N$) are the cash-flows of the liabilities (resp. the cash-flows of the N assets that make up the RP) at date t, one endeavors to determine a weight vector, where:

$$cf_{liab}(t) \approx \sum_{k=1}^{N} w_t^k . cf_{RP}^{k}(t),$$

In which w_t^k elements are constants. We will examine this approximation in its mathematical context in section 3.1, "Concerning the almost surely equality of the Least squares method".

2.4.2 Principle

The Replicating Portfolios (RP) technique enables to address various issues:

- The solvency analysis of the company in a context of prudential norms (e.g. Solvency II): the RP as a proxy of the best estimate of liabilities enables to assess the regulatory economic capital, to carry out alternative calculations of the requirements in terms of equity (in ORSA[1] for example), to determine the pricing of products taking into account the prudential environment, …
- The assessment of the sensitivities of a MCEV (Market Consistent Embedded value),
- The definition of hedging strategies, …

The reader may refer to Schrager (2008) for a description of the RP technique. The possible applications of this method are also mentioned in Algorithmics (2008) and Schrager (2008).

In this article we shall present an RP calibration method used to calculate the regulatory Solvency II capital. It should be mentioned that in order to address the issues listed above, complementary methods must be developed.

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[1] In the ORSA context (Own Risk and Solvency Assessment), the company develops its own definition of solvency.

We have seen above that the NS method is the best suited to Solvency II calculations as it enables to achieve a distribution of equity over one year while carrying out accurate simulations by means of risk-neutral secondary simulations. This method involves time consuming calculations. There are two possible ways of rendering the calculations of economic capital operational:

- Reducing the number of simulations in a NS application,
- Resorting to proxies associated with the items of the balance sheet.

In this article, we will not provide a detailed description of the acceleration methods for a NS projection. For more information on the subject, the reader may refer to Devineau and Loisel (2009).

The RP technique is part of the second category of methods that enable operational calculations of the company's requirements in terms of capital. It will be the subject of a presentation in this section.

In reality there are various ways in which this method may be implemented. Some companies use the RP technique in order to obtain the equity distribution at $t = 1$, others perform instantaneous stress tests in order to get a pseudo-distribution at the initial date. In this article we shall only consider the approaches that aim to determine the distribution of equity with a one year period.

In this type of application, "real world" simulations are carried out by means of an internal model for the first year. Then each simulation is used to assess the value of the RP and the equity. The RP can be valued by means of closed formulae or numeric methods (binomial or trinomial trees, Monte-Carlo techniques) depending on the complexity of the assets it includes.

We shall now introduce the following notations in order to formalise the calculations:

- A_1^p the market value of the asset at the end of the first period for the primary simulation p,
- L_1^p the best estimate of the liabilities at the end of the first period for the primary simulation p,
- Eq_1^p the shareholder's equity at the end of the first period for the primary simulation p,
- RP_1^p the value of RP at the end of the first period for the primary simulation p.

The items of the balance sheet at $t = 1$, for the primary simulation p are then calculated in the following manner:

$$L_1^p \approx RP_1^p,$$

And,

$$Eq_1^p = A_1^p - L_1^p \approx A_1^p - RP_1^p.$$

This application is illustrated by the following diagram:

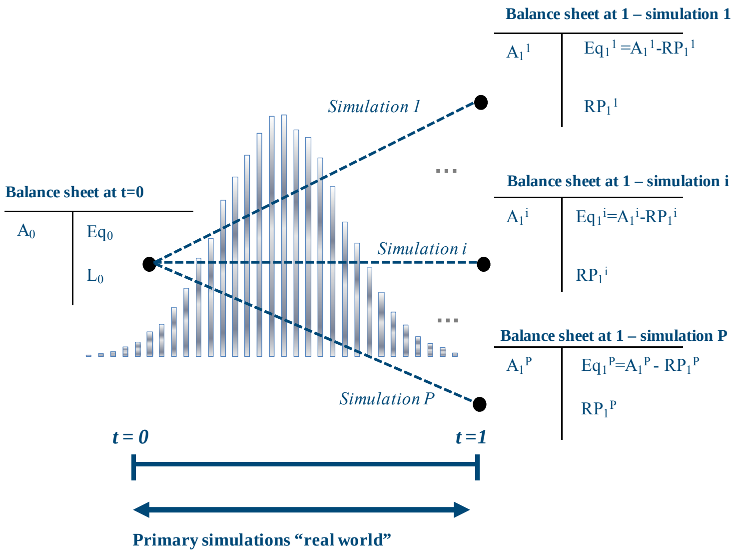

Figure 2: obtaining the distribution of equity with the RP method

One of the main advantages of the RP method is the significant size of the sample of Eq_1 thus achieved. This enables to guarantee the estimation robustness of the percentile assessment at 0.5% of the one-year equity.

However, the RP has to undergo thorough calibration as a bad RP/equity adjustment may give rise to erroneous estimations of the economic capital.

The alternative method that we have developed implies longer calibration times than the usual methods but it achieves a more robust estimation of the capital requirements. We will examine it in further depth in section 4 after a brief reminder of the calibration methods that are commonly used.

3. THE USUAL APPLICATIONS OF THE REPLICATING PORTFOLIOS TECHNIQUE

In this section we present the most commonly used methods.

3.1 Key ideas of standard approaches

Generally speaking, standard RPs follow this sequence:

- *Step 1:* construction of real-world or risk-neutral economic scenarios used for the calibration operations,
- *Step 2:* ALM calculations in order to obtain the liabilities cash flows for each date for all the simulations generated at Step 1,
- *Step 3:* selection of candidate assets and setting of their parameters,
- *Step 4:* determination of the make-up of the RP (i.e. estimation of the weight of each candidate asset) by means of an optimization program solving,
- *Step 5 :* goodness of fit,
- *Step 6 :* the RP is used as proxy of the liabilities' best estimate.

The reader may refer to Algorithmics (2008) for a description of the RP technique. In the following section we offer some comments pertaining to the aforementioned steps. The simulations (step 1) used for calibration purposes are often risk-neutral scenarios. However, some companies rely on real-world scenarios in order to achieve some degree of consistency between the calibration operations and the diffusion of the RP, as the latter necessarily implies real-world simulations. It can also be possible to give high weights to extreme scenarios to improve the robustness of the Replicating Portfolio in extreme economic conditions. The ALM projections (step 2) are generally carried out by means of an "MCEV"-type model that enables the valuing of the company's equity and liabilities at the date $t = 0$. The selection of the candidate assets that make up the RP (step 3) is based on expert judgment and does in no way result from an automatic and optimal process. In actual fact, there are several possible versions governing the resolution of the optimisation program; these will be described in further detail in the following section. In order to assess the goodness of fit (Step 5), companies perform graphical validations and work out the coefficient of determination (often called « R^2 ») on the sample of the pairs of actual values of the liabilities' cash flows and the RP. Adequacy tests such as Kolmogorov-Smirnov test, or stress tests (comparison of the value of the liabilities in a stressed economic condition and the Replicating Portfolio value in the same condition) are better tools to validate the fit. The RP is then projected in a real world environment in order to obtain a distribution of the best estimate of liabilities. The market value of the asset is also projected from the same set of scenarios in order to get by subtraction the distribution of the equity of the company and to estimate the amount of the Solvency II capital. It should be noted that the risk factors may be projected at $t = 1$ or $t = 0$ by means of instantaneous shocks[1].

In the following section we shall present a formalized version of the most commonly used calibration approaches.

3.2 Various calibration approaches

3.2.1 Introduction and notations

In practice there are two main types of methods: the replication of cash flows and the replication of the PVCF (present value of cash-flows). Although both are used for the calculation of the Solvency II economic capital, these approaches do not serve the exact same purpose. For example, a method based on the replication of cash flows may be used in order to address issues pertaining to the calculation of economic capital for various bankruptcy horizons. Indeed the replication of PVCF does not reflect the path dependency of liabilities because the replicated value is determined at one precise time. It proves to be very useful when used in an ORSA (Own Risk and Solvency Assessment) context. An approach that consists in replicating the PVFP enables to rapidly determine the sensitivities of a company's MCEV. For a presentation of standard approaches and their implementing, the reader may refer to Revelen (2009) and Larpin (2009).

We shall also mention the replication of terminal value, which we do not treat in this article.

We shall now introduce the following notations:

- T the liabilities' projection horizon,
- S the number of simulations required for calibration purposes,
- $cf_{liab}(s,t)$ the liability's cash-flow at date t in simulation *s*,
- $cf_{RP}^{k}(s,t)$ the cash flow resulting from the k[th] candidate asset of the RP at date *t* in simulation *s*, with $k = 1, \dots, N$,
- w_k the weight associated with the k[th] candidate asset of the RP,
- $L(0)$ (resp. $RP(0)$) the best estimate of the liabilities (resp. the value of the RP) at time 0

In the following sections, we provide a formalized version of the various calibration techniques listed above.

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[1] The instantaneous shocks are nevertheless homogenous to one year movements of the various drivers in a real-world context.

3.2.2 Replication of cash-flows

Schrager (2008) offers a description of this method and provides an example of application on a life insurance portfolio.

An approach to calibration based on cash flows replication enables to obtain, for each date, relatively close values of liability cash flows and RP cash flows[1]:

$$cf_{liab}(s,t) \approx \sum_{k=1}^{N} w_k \; cf_{RP}^{k}(s,t) \quad \forall s,t.$$

The most immediate technique to estimate the optimal weight vector $(w_1^*, \ldots, w_N^*)$, relies on an approach of the OLS-type (Ordinary Least Square) that leads to the solving of the following optimization program:

$$(w_1^*, \ldots, w_N^*) = \underset{(w_1, \ldots, w_N)}{\text{Argmin}} \sum_{t=1}^{T} \sum_{s=1}^{S} \left(cf_{liab}(s,t) - \sum_{k=1}^{N} w_k \; cf_{RP}^{k}(s,t) \right)^2 \qquad (P1)$$

In reality, this kind of criterion is quite rare. In practice, constraints are added to the optimization program in order to increase the calibration's consistency:

$$\begin{cases} (w_1^*, \ldots, w_N^*) = \underset{(w_1, \ldots, w_N)}{\text{Argmin}} \sum_{t=1}^{T} \sum_{s=1}^{S} \left(cf_{liab}(s,t) - \sum_{k=1}^{N} w_k \; cf_{RP}^{k}(s,t) \right)^2 \\ \textit{under the constraint} \; \dfrac{|L(0) - RP(0)|}{L(0)} \leq \varepsilon \end{cases} \qquad (P2)$$

Where ε is the level of error (determined by the user) between the liabilities' BE and the RP value at time $t = 0$.

The calculation of the Solvency II economic capital requires the valuing of the RP as it undergoes extreme stresses. Therefore, in order to provide greater robustness to the assessment of capital, some users introduce shock replication constraints in the calibration process:

[1] We shall return to the meaning of this equality further on in this article (see section 5.1).

$$\begin{cases} (w_1^*, \dots, w_N^*) = \underset{(w_1, \dots, w_N)}{\text{Argmin}} \sum_{t=1}^{T} \sum_{s=1}^{S} \left(cf_{liab}(s,t) - \sum_{k=1}^{N} w_k \; cf_{RP}^{k}(s,t) \right)^2 \\ \text{under the constraints :} \\ \dfrac{|L(0) - RP(0)|}{L(0)} \leq \varepsilon \\ \dfrac{|L_{shock\ 1}(0) - RP_{shock\ 1}(0)|}{L_{shock\ 1}(0)} \leq \varepsilon_1 \\ \dots \\ \dfrac{|L_{shock\ m}(0) - RP_{shock\ m}(0)|}{L_{shock\ m}(0)} \leq \varepsilon_m \end{cases} \qquad (P3)$$

Where $L_{shock\ i}(0)$ (resp. $RP_{shock\ i}(0)$) represents the value at $t = 0$ of the BE of the liabilities (resp. of the RP) conditional on shock i and where ε_i is the acceptable level of error for this shock.

Since liabilities are projected over a fairly long period (between 30 and 50 years) it may be advisable to group cash flows into broadly defined periods that are often called "time buckets". Therefore, in most cases, the optimization programs aim to replicate cash flow aggregated for each "time bucket ".

3.2.3 The replication of the PVCF

If $\delta(s,t)$ is the discount factor's value[1] at time *t* in simulation s, and considering the PVCF (Present Value of Cash-Flows) for liabilities and for the RP in the simulation s, then:

$$PVCF_{liab}(s) = \sum_{t=1}^{T} \delta(s,t) \cdot cf_{liab}(s,t),$$

And,

$$PVCF_{RP}^{(w_1,\dots,w_N)}(s) = \sum_{t=1}^{T} \delta(s,t) \cdot \sum_{k=1}^{N} w_k \; cf_{RP}^{k}(s,t) \,.$$

The most common optimization program for the calibration of the RP is similar to the last criterion mentioned in the previous section:

[1] This is the factor that makes martingale discounted prices.

$$\begin{cases} (w_1^*, \dots, w_N^*) = \underset{(w_1, \dots, w_N)}{\text{Argmin}} \sum_{s=1}^{S} \left(PVCF_{liab}(s) - PVCF_{RP}^{(w_1, \dots, w_N)}(s) \right)^2 \\ \text{under the constraints :} \\ \frac{|L(0) - RP(0)|}{L(0)} \leq \varepsilon \\ \frac{|L_{shock\ 1}(0) - RP_{shock\ 1}(0)|}{L_{choc\ 1}(0)} \leq \varepsilon_1 \\ \dots \\ \frac{|L_{shock\ m}(0) - RP_{shock\ m}(0)|}{L_{shock\ m}(0)} \leq \varepsilon_m \end{cases} \qquad (P4)$$

Note: the implementation of an OLS approach (without adding any constraints) necessarily leads to the equality between the BE value and that of the RP at $t = 0$. In a standard linear regression the expected values of the response variable are equal to those of the predictor, which leads, under a risk-neutral Q probability, to the following relation:

$$E_Q(PVCF_{liab}) = E_Q\left(PVCF_{RP}^{(w_1^*, \dots, w_N^*)}\right)$$
$$\Leftrightarrow L(0) = RP(0).$$

The integration of shock replication constraints induces a loss of this property. It is therefore necessary to add it to the set of constraints of the optimization program.

3.3 Replication of liabilities or replication of profits?

As we explained in the previous sections, the operational approaches developed by companies that use the RP technique for the calculation of their economic capital rely on the replication of liabilities (cash flows or present value of cash flows).

However, in certain cases it is more accurate to directly replicate the company's profits (in cash flows or in PVFP) according to the methods described above. This choice often proves to be the wisest because the replication of profits allows for greater control over the error committed in the capital assessment. A very small deviation in the replication of liabilities can lead to significant differences in terms of SCR because the size of the liabilities' BE is much greater than that of the company's equity.

For these reasons, the method that we developed and that is presented in the following section relies on the replication of the company's equity. In the part pertaining to applications we shall endeavor to present the issues relating to the choice of which variable to replicate and we will demonstrate that it's far more accurate to consider shareholder's equity.

Two explanations are nevertheless put forward to justify "liability replication" approaches:

- The optional analysis of insurance contracts simplifies the choice of candidate RP assets and the setting of their parameters,
- This technique enables a "line by line" projection of the company's assets.

As we mentioned in section 2.4.1, the first argument is hardly valid in practice because the replication of economic variables calculated by means of an internal model is not operationally achievable with techniques developed in the field of quantitative finance.

Furthermore, a "line by line" projection of the market value of the company's assets can lead to some mismatch in the economic capital calculation process. Indeed, in most internal models the ALM projections are carried out on aggregated asset data (in order to reduce calculation times). Therefore, the sample on which the RP is calibrated is determined on an "aggregated basis" and not on a "line by line" basis. Inconsistencies due to the heterogeneity of the processes may occur when calculating economic capital on the basis of an estimation of equity that stems from the difference between the value of the asset in "line by line" mode and the value of the calibrated RP in "aggregated" mode.

4. ALTERNATIVE CALIBRATION METHOD

4.1 Introduction

The technique that we have developed and which we present in this section is a calibration of the RP with a sub-sample of equity values at t=1. This method also enables to automatically select candidate assets that make up the RP and to set their parameters in an optimal way.

In the above section we described how usual methods of RP calibration consist in minimizing the squared differences of the following type:

$$\sum_{t=1}^{T}\sum_{s=1}^{S}\left(cf_{liab}(s,t)-\sum_{k=1}^{N}w_k\ cf_{RP}^{k}(s,t)\right)^2,$$

Or,

$$\sum_{s=1}^{S}\left(PVFC_{liab}(s)-PVFC_{RP}^{(w_1,\dots,w_N)}(s)\right)^2.$$

However, our approach relies on a calibration that requires the knowledge of the equity results after one year. The elements taken into consideration are therefore homogeneous with Best Estimates and not with cash flows. The process therefore consists in minimizing deviations of the following type:

$$\sum_{p} \left(Eq_p - RP_p^{(w_1,\ldots,w_N)} \right)^2,$$

Where $RP_p^{(w_1,\ldots,w_N)}$ is the RP value at $t = 1$ in the simulation p.

Contrary to standard approaches that rely on minimizations of cash-flow differences, the items Eq_p and $RP_p^{(w_1,\ldots,w_N)}$ considered in our proposed alternative method presented herein correspond to prices and are therefore consistent with the (conditional) expected values of the cash flows.

4.2 The key concepts of the method

The use of a parametric form usually yields highly satisfactory results for the assessment of the economic capital (see Devineau and Loisel (2009b)). Therefore, assuming that each term of the parametric form can be replicated by a sub-replicating portfolio, it becomes possible to achieve a satisfactory global RP for the estimation of the economic capital.

Consider the n-uplet of risk factors $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ that represents the hazards summarizing the intensity of the risk in each primary simulation and consider the following parametric form:

$$f(\varepsilon_1, \ldots, \varepsilon_n) = \sum_{(i_1,\ldots,i_n)\epsilon E} A_{i_1,\ldots,i_n} \varepsilon_1^{i_1} \ldots \varepsilon_n^{i_n},$$

Where E is a subset[1] of $\{(i_1, \ldots, i_n)\epsilon\mathbb{N}^n /: \ i_1 + \cdots + i_n \leq d\}$.
$Eq_{param} = f(\varepsilon_1, \ldots, \varepsilon_n)$ is the random variable "shareholder's equity estimated with the parametric form".

The alternative method consists in adding a RP that we shall write $RP^{(i_1,\ldots,i_n)}$ to each term $\varepsilon_1^{i_1} \ldots \varepsilon_n^{i_n}$ of the parametric form.

By triangle inequality, we obtain:

$$\|Eq - RP\| \leq \|Eq - Eq_{param}\| + \|Eq_{param} - RP\|$$

$$\|Eq - RP\| \leq \|Eq - Eq_{param}\| + \sum_{(i_1,\ldots,i_n)\epsilon E} A_{i_1,\ldots,i_n} \left\|\varepsilon_1^{i_1} \ldots \varepsilon_n^{i_n} - RP^{(i_1,\ldots,i_n)}\right\|,$$

[1] The E subset is a choice of regressors among elements $\varepsilon_1^{i_1} \ldots \varepsilon_n^{i_n}$.

Where Eq (resp. RP) is the value of the equity (resp. of the Replicating Portfolio) at $t = 1$.

The above inequality shows that an accurate replication of each term enables to achieve a RP that satisfactorily replicates the Eq variable. This constitutes an upper bound of the error; we shall demonstrate how the alternative method often leads to stronger adjustments than the methods of the “parametric form” kind.

The alternative method follows this sequence:

- *Step 0:* construction of real-world economic scenarios and preliminary selection of adverse simulations,
- *Step 1:* calculation of the shareholder’s equity value (at t = 1) relating to the previously selected scenarios and use of a parametric form for the analysis of the convexity of the "equity curve".
- *Step 2:* construction of replicating sub-portfolios which fit each term of the parametric form.
- *Step 3:* regression of the shareholder’s equity distribution at t= 1 on the price of candidate assets in order to obtain the asset mix (i.e. the weight vector) of the RP,
- *Step 4:* projection at $t = 1$ of the RP on a complete set of real world simulations and deduction of the economic capital.

In the following section we shall review in further detail the various steps given above.

The use of a parametric form (Step 1) enables to study the convexity of the equity function on risk factors. This preliminary study requires knowledge of a limited number of equity values at $\mathrm{t} = 1$ (Step 0). Once the elements of the parametric form are known, candidate assets are automatically deduced (Step 2) and their parameters are set (determination of underlying maturities, the moneyness levels and the exercise dates of the assets being considered) in an optimal manner during the sub-replication phase.

The composition (asset mix) of the RP is then estimated (Step 3) by linear regression (of the sub-sample of the equity on prices at $\mathrm{t} = 1$ of RP assets). The market value of the RP is then simulated (Step 4) over a one-year period on the basis of a complete set of real-world simulations so as to obtain a sample of the equity variable. The calculation of the economic capital is then instantaneous.

In the following section, we provide a more detailed description of the steps involved in the alternative method. For the purpose of clarity we will only consider two risks: the level risk on the "stock" index and the level risk on interest rates. We shall adopt the following notations: ε_S (resp. ε_{ZC}) as the stock (resp. interest rates) risk factor. However, this approach may be broadened to a higher dimension.

4.3 Calculation of a sample of equity values at t=1

The calibration of the parametric form and of the asset mix of the RP requires a set of equity values at $t = 1$. In order to guarantee the tractability of the method, it is crucial to consider a number of equity values as little as possible. Indeed, the determination of each of these values relies on Monte-Carlo simulations and on conditional risk-neutral tables taking into account the first period information.

Hence, for a fixed primary simulation p, the shareholder's equity is estimated in the following manner:

$$Eq_1^p \approx R_1^p + \frac{1}{S}\sum_{s=1}^{S}\sum_{u\geq 2}\frac{\delta_u^{p,s}}{\delta_1^p}R_u^{p,s},$$

Where,

- $R_u^{p,s}$ the profit at time $u > 1$ for the primary simulation $p \in \{1, \dots, P\}$ and secondary simulation $s \in \{1, \dots, S\}$,
- $R_u^{p,s}$ the profit of the first period fort the primary simulation p,
- $\delta_u^{p,s}$ the discount factor of the period $u > 1$ for the primary simulation p and the secondary simulation s,
- δ_u^p the discount factor of the first period for the primary simulation p,
- Eq_1^p the shareholder's equity at the end of the first period for the primary simulation p.

In order to reduce the equity calculations to a minimum, the primary simulations are chosen according to their degree of adversity. In order to achieve this, one considers a norm criterion. As a reminder, it should be noted that the norm of the primary scenario p is calculated in the following manner:

$$\left\|\left(\varepsilon_S^p, \varepsilon_{ZC}^p\right)\right\|_\rho \triangleq \sqrt{\left(\varepsilon_S^p\right)^2 + \left(\varepsilon_{ZC}^p\right)^2 - 2\rho_{\varepsilon_S,\varepsilon_{ZC}}\varepsilon_S^p\varepsilon_{ZC}^p},$$

Where ε_S (resp. ε_{ZC}) represents the stock risk factor (resp. rates) in simulation p and $\rho_{\varepsilon_S,\varepsilon_{ZC}}$ is the linear correlation between the two factors. We can point out two possible cases. Either risk factors are known because they are inputs of the economic scenarios (representing the hazard projected in the stock index or in the zero-coupon bond prices), or they can be extracted from the economic scenarios. In the second case, we consider that risk factors are calculated as the return of each driver that we standardize and normalize.

The value of equity $\left(\mathrm{Eq}_1^p\right)_{p=1,\dots,N}$ relating to N scenarios of worst norms is then established. It should be noted that these scenarios correspond to the extreme values of the "equity" variable. This selection leads to a more robust calibration of the RP used to calculate the economic capital.

4.4 Analysis of convexity with a parametric form

The aim of this step is to fit a parametric form using the set of N equity values $\left(Eq_1^p\right)_{p=1,\dots,N}$ obtained previously. The make-up of the parametric form enables to study the analytical properties of the "equity" function and constitutes a very efficient tool for the selection and setting of the parameters of the RP's assets. The calibration of the parametric form relies on the minimization of the following optimization program[1]:

$$\left(A^*_{i_S,i_{ZC}}\right)_{(i_S,i_{ZC})\in E} = \underset{\left(A_{i_S,i_{ZC}}\right)_{(i_S,i_{ZC})\in E}}{\text{Argmin}} \sum_{p=1}^{N}\left(Eq_1^p - \sum_{(i_S,i_{ZC})\in E} A_{i_S,i_{ZC}}\varepsilon_S^{i_S}\varepsilon_{ZC}^{i_{ZC}}\right)^2,$$

Under the constraints,

$$\frac{\left|SCR^S_{param} - SCR^S\right|}{SCR^S} \le e_S,$$

And,

$$\frac{\left|SCR^{ZC}_{param} - SCR^{ZC}\right|}{SCR^{ZC}} \le e_{ZC},$$

Where SCR^S_{param} (resp. SCR^{ZC}_{param}) is the stock (resp. interest rates) SCR calculated with the marginal stock parametric form (resp. interest rates) and where SCR^S (resp. SCR^{ZC}) is the stock (resp. interest rates) SCR that results from the model considering for the one-year scenario the stock (resp. interest rates) 0.5% percentile. The Terms e_S and e_{ZC} correspond to the maximum errors allowed in the replication of marginal capitals by means of the parametric form.

[1] The integration of constraints leads us to use numerical methods to solve the optimization program. The same comment is true for all optimization programs we have to solve during the process

This application achieves greater robustness in terms of fitting in extreme simulations of the model's results and the results produced by the parametric form.

It should also be noted that in order to avoid issues related to over-parameterization, the smallest number of regressors possible should be considered. In practice, a degree of $d \leq 3$ and a careful selection of covariates offer good results.

As an example, we provide here the structure of the parametric form of the savings product studied in part 6:

$$Eq_{param} = A_{0,0} + A_{1,0} \cdot \varepsilon_S + A_{2,0} \cdot \varepsilon_S^2 + A_{3,0} \cdot \varepsilon_S^3 + A_{0,1} \cdot \varepsilon_{ZC} + A_{0,2} \cdot \varepsilon_{ZC}^2 + A_{1,1} \cdot \varepsilon_S \cdot \varepsilon_{ZC}$$

The cube of the interest rates risk factor is not significant and the only crossed term that is retained is the 1st order one.

4.5 Choice of candidate assets and setting of parameters

This step consists in associating each term of the parametric form with a sub-RP optimally fixed.

One should remember that one of the main disadvantages of the usual methods is that the choice of candidate assets relies entirely on expert judgment and does not result from an automatic method.

However, the alternative method enables to assign a sub RP, written $RP^{(i_S,i_{ZC})}$, to each term $\varepsilon_S^{i_S} \cdot \varepsilon_{ZC}^{i_{ZC}}$ of the parametric form by resolving an optimization program.

4.5.1 Introduction

In this section we offer an introduction to the principle of sub replication of the terms of the parametric form.

In order to achieve this, one should examine the first degree interest rates term ε_{ZC} and consider a Taylor series of the first order of a zero-coupon bond price. If:

$$P(1,m) \approx drift_m^{ZC} + \sigma_m^{ZC} \cdot \varepsilon_{ZC},$$

Where $P(1,m)$ is the price at $t = 1$ of a zero-coupon bond with a time to maturity of m and $drift_m^{ZC}$ and σ_m^{ZC} some constant variables related to the log efficiency of the ZC's price.

It is easy to draw an expression of the rate risk factor from the ZC's price:

$$\varepsilon_{ZC} \approx \frac{1}{\sigma_m^{ZC}} P(1,m) - \frac{drift_m^{ZC}}{\sigma_m^{ZC}}$$

The right-hand term of the above equation uses two components: a ZC component and a cash component.

Let us define I_t^{cash} the value at t of a risk free asset and we assume that $I_0^{cash} = 1$ and $I_1^{cash} = 1/P(0,1)$.

The portfolio $RP_{ZC} = w_{cash} I_1^{cash} + w_{ZC} P(1,m)$ of weight,

$$w_{ZC} = \frac{1}{\sigma_m^{ZC}},$$

And,

$$w_{cash} = -\frac{drift_m^{ZC}}{\sigma_m^{ZC}} \cdot P(0,1),$$

Enables an accurate replication of the ε_{ZC} factor.

4.5.2 The general principle of sub-replication

The analyses presented above reveal that the ε_{ZC} factor may be replicated with ZC of different maturities. However, it is crucial to determine which maturity is optimal for the replication of that term. In order to achieve this, the following optimization program must be solved:

$$(w_{cash}^*, w_{ZC}^*, m^*) = \underset{(w_{cash}, w_{ZC}, m\ )}{\text{Argmin}} \sum_{p=1}^{P} \left(\varepsilon_{ZC}^p - (w_{ZC} \cdot P^p(1,m) + w_{cash} \cdot I_1^{cash})\right)^2,$$

Where P is the total number of primary simulations considered for the calculation of economic capital (therefore $P \gg N$) and $P^p(1,m)$ is the price of a ZC with a time to maturity of m in the primary simulation p.

This optimization provides the weight of assets included in the sub-RP of the ε_{ZC} term, as well as their parameters (the only parameter that is to be determined is the maturity of the ZC chosen for the replication).

Similarly, once the list of candidate assets is chosen for the replication of the $\varepsilon_S^{i_S} \cdot \varepsilon_{ZC}^{i_{ZC}}$ factor, the parameters are assessed by minimization of the criterion:

$$\theta^* = \underset{\theta}{\text{Argmin}} \sum_{p=1}^{P} \left(\left(\varepsilon_S^p\right)^{i_S} \cdot \left(\varepsilon_{ZC}^p\right)^{i_{ZC}} - RP_\theta^{(i_S, i_{ZC})} \right)^2,$$

Where $RP_\theta^{(i_S, i_{ZC})}$ is the sub-RP that enables to replicate the $\varepsilon_S^{i_S} \cdot \varepsilon_{ZC}^{i_{ZC}}$ term and θ is the weight and parameters’ vector for the candidate assets being considered.

The only information needed for this application is the risk factor values and the prices of candidate assets in each primary simulation. These calculations can often be carried out outside the internal model. In order not to compromise the tractability of the sub-replication phase, the candidate assets that require valuation (e.g. derivatives) are valued by means of closed formulae. However, during the determination of the global asset mix of the RP we favor a valuation with the Monte-Carlo technique (see hereunder).

4.5.3 The choice of candidate assets according to the term that is to be replicated

The selection of candidate assets depends on the level of convexity of the term that needs to be replicated.

For the replication of a fist degree term, one resorts directly to the corresponding underlying asset (we have seen that the term ε_{ZC} is replicated by means of cash and a ZC). However, in order to replicate the term ε_S, it often becomes necessary to consider, in addition to the "cash" and "stock index" components, the derivatives (in order to adjust the convexity when very high volatility prevents the use of a fist order Taylor development).

Intuitively, the replication of second order terms implies, in addition to the underlying assets, the derivatives (based on stock or interest rates, depending on the term being considered.).

The following table lists the types of candidate assets that make up the sub-RP for each one of the terms of the parametric form (with the exception of crossed effects):

Terms	**Assets per sub-RP**
ε_{ZC}	Cash, ZC
$\varepsilon_{ZC}^2, \varepsilon_{ZC}^3$	Cash, ZC, caplets, floorlets
ε_S , $\varepsilon_S^2, \varepsilon_S^3$	Cash, stock index, calls, puts

Table 1: mapping of the terms of the parametric form with the assets of the sub-RP

Note: the parametric form contains crossed terms (i.e. factors of the $\varepsilon_S^{i_S} \cdot \varepsilon_{ZC}^{i_{ZC}}$ type with $i_S > 0$ and $i_{ZC} > 0$). Therefore, direct replication may prove to be difficult and requires the use of hybrids assets (e.g. convertible bonds). However, the prices of stock derivatives are equal to expected values of payoffs depending both on interest rates and stock index ; thus these assets enable to catch indirectly the crossed effects. We shall therefore not try to replicate specifically these factors and we shall base ourselves on the assumption that some candidate assets of the RP take them into account implicitly.

4.5.4 Examples of sub-replication

In this section we present as an example the minimization criteria for the replication of terms ε_S and ε_{ZC}^2.

Denote :

- $P^p(1,m)$ the price of a zero-coupon at $t=1$ with a time to maturity of m for the primary simulation p,
- S_1^p the stock index at $t=1$ for the primary simulation p,
- $C^p(K,m)$ (resp. $P^p(K,m)$) the price of a call[1] (resp. put) with an exercise date *m* and a strike *K* for the primary simulation p,
- $Caplet^p(K,m,t)$(resp. $Floorlet^p(K,m,t)$) the price of a caplet (resp. floorlet) with an exercise date *m*, a strike rate *K* and an underlying interest rate with a time to maturity *t* for the primary simulation *p*.

We shall consider the portfolios given below in order to replicate respectively ε_S and ε_{ZC}^2:

$$RP^p{}^{(1,0)}_{\theta_{(1,0)}} = w_{cash}^{(1,0)} I_1^{cash} + w_S S_1^p + w_{C_1} C^p(K_1,m_1) + w_{C_2} C^p(K_2,m_2) + w_{P_1} P^p(K_3,m_3)$$

$$w_{P_2} P^p(K_4,m_4),$$

And,

$$RP^p{}^{(0,2)}_{\theta_{(0,2)}} = w_{cash}^{(0,2)} I_1^{cash} + w_{ZC} P^p(1,m_{ZC}) + w_{Cplt_1} Caplet^p(K_1,m_1,t_1) + w_{Cplt_2} Caplet^p(K_2,m_2,t_2)$$

$$+ w_{Flt_1} Floorlet^p(K_3,m_3,t_3) + w_{Flt_2} Floorlet^p(K_4,m_4,t_4).$$

The vectors of parameters $\theta_{(1,0)}$ and $\theta_{(0,2)}$ are determined by solving the following optimization programs:

$$\theta^*{}_{(1,0)} = \underset{\theta_{(1,0)}}{\text{Argmin}} \sum_{p=1}^{P} \left(\varepsilon_S^p - RP^p{}^{(1,0)}_{\theta_{(1,0)}}\right)^2,$$

And,

$$\theta^*{}_{(0,2)} = \underset{\theta_{(0,2)}}{\text{Argmin}} \sum_{p=1}^{P} \left(\left(\varepsilon_{ZC}^p\right)^2 - RP^p{}^{(0,2)}_{\theta_{(0,2)}}\right)^2.$$

[1] The implied stock (resp. swaptions) volatility is assumed to be known and equal to the volatilities observed at $t=0$.

Comments:

- In practice, it is sufficient to consider only two calls, puts (resp. caplets, floorlets) for a satisfactory replication of terms ε_S (resp. ε_{ZC}^2). This prevents over-parameterization issues. If the calibration is performed with a limited number of primary simulations (i.e. a small *N* value) and implies a large number of candidate assets, it may lead to a heavily compromised predictive quality of the portfolio.
- The replication of terms of the third degree does not give rise to any particular problems and may be treated as second degree terms by jointly introducing long and short positions.
- The integration of certain financial risk factors in the parametric form and the RP may sometimes be delicate[1] (this is the case for stock and interest rates volatilities).

4.6 Calibration of the Replicating Portfolio asset mix

Once the candidate assets are chosen and their parameters are set, the sub-RPs are consolidated in order to obtain a global portfolio. The weight of each asset is then determined in this last step by minimizing a target function of the aggregated portfolio.

If:

- M is the number of assets included in the RP,
- $A_k^p(t)$ is the market value at date t of asset k for a primary simulation p,
- $(w_k^*)_{k=1,\dots,M}$ is the optimal weight vector.

Then the optimization program that is to be solved is as follows:

$$(w_k^*)_{k=1,\dots,M} = \underset{(w_k)_{k=1,\dots,M}}{Argmin} \sum_{p=1}^{N} \left(Eq_1^p - \sum_{k=1}^{M} w_k \, A_k^p(1) \right)^2 ,$$

Under the constraint,

$$\frac{\left| Eq_0 - \sum_{k=1}^{M} w_k \, A_k(0) \right|}{Eq_0} \leq e,$$

Where the term e is the level of error accepted by the user in the replication of equity at $\mathrm{t} = 0$.

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[1] However, it is possible to use fake assets for the replication of these factors.

The adding of a constraint provides greater robustness to the calibration of the RP for the purpose of calculating the economic capital.

Once the determination of the weight is completed, the one-year value of the RP is simulated with the P primary simulations included in the complete set of real-world scenarios.

As described above, the market value of the assets are determined with the Monte-Carlo technique on risk-neutral tables which depend on the first period information. This application ensures greater consistency with the assessment method of items of the balance sheet with the internal model (Monte-Carlo type pricing). Furthermore, a Monte-Carlo valuation is universal and does not depend on the financial models that are used, as is the case in an approach based on closed formulae.

Finally the economic capital is calculated on the basis of the distribution of the RP's value at $t=1$ by approximation:

$$SCR = Eq_0 - P(0,1).RP_1^{[0.5\%.P]},$$

Where $RP_1^{[0.5\%.P]}$ represents the $[0.5\%.P]^{th}$ worst value of the distribution thus achieved.

Comments:

- The aim of the method presented here is to replicate equity at time 1. In the case of a company that doesn't have a one-year projection model, it is still possible to implement this method by means of instantaneous shocks at $t=1$.
- This method may also be used to replicate the best estimate of liabilities. In sections 5 and 6, we shall see how even a satisfactory replication of liabilities can lead to significant deviations in terms of economic capital.

5. THEORETICAL ANALYSIS OF THE VARIOUS METHODS

The aim of this section is to shed some theoretical light on the usual calibration techniques for RPs and to highlight a number of points that can cause these approaches to fail.

5.1 Concerning the "almost surely" equality of the Least squares method

In the technical literature devoted to the RP method there is an implicit fundamental assumption of the following “almost surely” equality:

$$cf_{liab}(t) = cf_{RP}(t) \quad a.s$$

Indeed, an "almost surely" equality between the liabilities' cash flows variable and the RP's cash flows variable enables, whatever the considered calculation data or the probability environment may be (real-world or risk-neutral), to obtain an equality of the best estimate of the liabilities and that of the RP. In these circumstances, the RP is a very powerful calculation proxy that can be used beyond the realm of regulatory economic capital calculations to address issues pertaining to ORSA, ERM...

However, given the real complexity of the liabilities, this equality cannot be verified and the method is a simple linear regression of the target variables $cf_{liab}(t)$ on the covariates $\left(cf_{RP}^{k}(t)\right)_{k=1,\dots,N}$ that correspond to the cash flows of the *N* assets included in the RP.

For the purpose of clarity, one will assume hereunder without loss of generality that the aim is to replicate the liability cash-flows of a single period *t*.

By using the first optimization program presented in section 3.2.2, the matching of the "almost surely" equality type,

$$cf_{liab}(t) = \sum_{k=1}^{N} w_k \ cf_{RP}^{k}(t) \quad a.s,$$

Becomes a matching of the "linear regression" type,

$$cf_{liab}(t) = \sum_{k=1}^{N} w_k \ cf_{RP}^{k}(t) + U(t),$$

With $E_Q\big(U(t)\big) = 0$ and $E_Q\left(cf_{RP}^{k}(t).U(t)\right) = 0 \,,\ k = 1, \dots, N.$

If $L_t(0)$ (resp. $RP_t(0)$) is the best estimate (resp. the market value) at t = 0 associated with the liabilities' cash flows (resp. the RP) for period t, one obtains:

$$L_t(0) = E_Q\left(\delta(t).cf_{passifs}(t)\right) = E_Q\left(\delta(t).\sum_{k=1}^{N} w_k \ cf_{RP}^{k}(t) + \delta(t).U(t)\right)$$

$$= \left(\sum_{k=1}^{N} w_k \ E_Q\left(\delta(t).cf_{RP}^{k}(t)\right)\right) + E_Q\big(\delta(t).U(t)\big)$$

$$= RP_t(0) + E_Q\big(\delta(t).U(t)\big).$$

It should be noted that in order to match $L_t(0)$ and $RP_t(0)$, the variables $\delta(t)$ and $U(t)$ must be uncorrelated.

This property is not guaranteed and, as explained above, in order to satisfy the equality, it is generally added as a constraint in the optimization program.

The apparition of noise $U(t)$ inherent to the loss of the "almost surely" equality can have significant effects on the calculation of the economic capital. Two major issues arise:

- The change of probability: the calculation of the economic capital relies on equity simulations in a real-world universe at $t = 1$[1]. However, when the calibration is carried out under a risk-neutral probability, the properties of the linear regression (and particularly the nullity of the residuals' expectations) can be unstable from one universe to the next.
- The assessment of extreme percentiles: for central "scenarios" a very low perturbation caused by the residual may be expected (because $E_Q\big(U(t)\big) = 0$), however, for the assessment of the extreme percentiles, the impact can be significant.

These two problems can have a strong negative impact on the robustness of the estimation of the Solvency II economic capital. They will be examined in further detail in the following sections.

5.2 The change of probabilities

5.2.1 Introduction

It should be noted that the calibration of the RP is often carried out in a risk-neutral universe whereas the calculation of the economic capital is based on a "real-world" projection of the RP.

With the notations presented above, the values at $t = 1$ of the liabilities $L(1)$ and that of the Replicating Portfolio $RP(1)$ given the "real-world" information of the first period F_1^{RW}, are calculated as follows:

$$L(1) = E_Q\left[\sum_{t\geq 2} \frac{\delta(t)}{\delta(1)} cf_{liab}(t) \middle| F_1^{RW}\right],$$

$$RP(1) = E_Q\left[\sum_{t\geq 2} \frac{\delta(t)}{\delta(1)} cf_{RP}(t) \middle| F_1^{RW}\right].$$

« **Real world** » information for the first year

Risk neutral universe calibration

As we explained above, the "real-world" treatment of the first period can considerably modify the goodness of fit between the RP and the liabilities examined in a

[1] The simulations are sometimes carried out for operational reasons at $t = 0$. However the instantaneous shocks are deemed to be homogenous with one year movements of financial drivers in a real-world environment.

risk-neutral environment.

5.2.2 Theoretical analysis

The process does not necessarily ensure the stability of the multiple linear regression method on which the RP technique is based. Therefore, even though the adjustment of the cash flows of the RP and the cash flows of the liabilities seems satisfactory at $t = 0$, its replication may be compromised by conditioning with the real-world information of the first period.

The equality of the best estimate of the liabilities and the value of the RP is not stable when changing to conditional expectation:

$$L_t(1) = E_Q(\delta_1(t).cf_{liab}(t)|F_1^{RW})$$
$$= E_Q(\delta_1(t).cf_{RP}(t) + \delta_1(t).U(t)|F_1^{RW})$$
$$= E_Q(\delta_1(t).cf_{RP}(t)|F_1^{RW}) + E_Q(\delta_1(t).U(t)|F_1^{RW})$$
$$= RP_t(1) + E_Q(\delta_1(t).U(t)|F_1^{RW}),$$

With,

$$\delta_1(t) = \frac{\delta(t)}{\delta(1)}.$$

In order to satisfy the equality between $L_t(1)$ and $RP_t(1)$, one must have: $E_Q(\delta_1(t).U(t)|F_1^{RW}) = 0$.

A condition to satisfy this equality is the following:

- the variables $\delta_1(t)|F_1^{RW}$ and $U(t)|F_1^{RW}$ are uncorrelated,
- the conditional expectation of the noise is equal to *0* : $E_Q(U(t)|F_1^{RW}) = 0$.

The behavior of $E_Q(\delta_1(t).U(t)|F_1^{RW})$ in extreme scenarios can therefore have a very negative effect on the robustness of the assessment of economic capital.

Comments:

- The alternative method that we have implemented enables to address this issue as the calibration process relies on shareholder's equity values calculated in extreme scenarios.
- Our method also takes structurally into account the calculation timing that result from the definition of the Solvency II capital, in which the initial equity is compared with the extreme values of equity at $t = 1$. Indeed, the

determination of the asset mix relies on a minimization of the differences between the value of the RP and the equity after one year, whereas the techniques that are usually used are based on elements projected at the initial date. This may give rise to certain distortions in the assessment of the economic capital.

5.3 Replication of liabilities or replication of profits?

As specified above, the usual applications of RPs are systematically aimed at replicating the distribution of the best estimate of liabilities. However, a slight replication error at the level of the liabilities can lead to very weak estimations of the economic capital. In this section we propose a formalization of this problematic.

Let us analyze the error of assessment that occurs in a liability replication method. If *p* is the primary scenario that corresponds to the 0.5% percentile of variable Eq(1), as estimated with the RP technique, then:

$$\mathrm{C_{RP}} = \widehat{\mathrm{Eq}_0} - \mathrm{P}(0,1).\left(\mathrm{A_p}(1) - \mathrm{RP_p}(1)\right).$$

The deviation between the two amounts of capital is as follows:

$$\frac{|\mathrm{C_{RP}} - \mathrm{C}|}{\mathrm{C}} = \frac{\left|\widehat{\mathrm{Eq}_0} - \mathrm{P}(0,1).\left(\mathrm{A_p}(1) - \mathrm{RP_p}(1)\right) - \mathrm{Eq}_0 + \mathrm{P}(0,1).\,\mathrm{q}_{0.5\%}(\mathrm{Eq}_1)\right|}{\mathrm{C}}$$

Assuming that $\widehat{\mathrm{Eq}_0} \approx \mathrm{Eq}_0$, then:

$$\frac{|\mathrm{C_{RP}} - \mathrm{C}|}{\mathrm{C}}$$

$$\approx \mathrm{P}(0,1) \cdot \frac{\left|\left(\mathrm{A_p}(1) - \mathrm{RP_p}(1)\right) - \mathrm{q}_{0.5\%}(\mathrm{Eq}_1)\right|}{\mathrm{C}}$$

$$= \mathrm{P}(0,1) \cdot \frac{\left|\left(\mathrm{A_p}(1) - \mathrm{RP_p}(1)\right) - E\left[\mathrm{A}(1)|\mathrm{Eq}(1) = \mathrm{q}_{0.5\%}(\mathrm{Eq}(1))\right] + E\left[\mathrm{L}(1)|\mathrm{Eq}(1) = \mathrm{q}_{0.5\%}(\mathrm{Eq}(1))\right]\right|}{\mathrm{C}}$$

Under the assumption

$\mathrm{A_p}(1) \approx E\left[\mathrm{A}(1)|\mathrm{Eq}(1) = \mathrm{q}_{0.5\%}(\mathrm{Eq}(1))\right]$, therefore:

$$\frac{|\mathrm{C_{RP}} - \mathrm{C}|}{\mathrm{C}} \approx \frac{\left|\mathrm{RP_p}(1) - E\left[\mathrm{L}(1)|\mathrm{Eq}(1) = \mathrm{q}_{0.5\%}(\mathrm{Eq}(1))\right]\right|}{L(0)} \cdot \frac{L(0) \cdot \mathrm{P}(0,1)}{\mathrm{Eq}_0 - \mathrm{P}(0,1).\,\mathrm{q}_{0.5\%}(\mathrm{Eq}_1)}$$

The above relation relies on two terms: the first one is the liability replication error and the second one is an amplification coefficient, written *M*:

$$\mathrm{M} = \frac{L(0) \cdot \mathrm{P}(0,1)}{\mathrm{Eq}_0 - \mathrm{P}(0,1).\,\mathrm{q}_{0.5\%}(\mathrm{Eq}_1)}.$$

If the company has sufficient equity to maintain solvency levels over one year with a level of 99.5%, the $q_{0.5\%}(Eq_1)$ amount is positive[1] and one then obtains:

$$M = \frac{L(0) \cdot P(0,1)}{Eq_0 - P(0,1).\, q_{0.5\%}(Eq_1)} \geq \frac{L(0) \cdot P(0,1)}{Eq_0}.$$

The ratio between "Best Estimate and equity" depends on the company, but as an example, for life-insurance portfolios on the French market as of the 12/31/2008 it is often between 15 and 40.

In this case, the error of assessment of economic capital is at least 15 times bigger than the error of liability replication. In other words, a replication error of 1% logically induces an error of capital assessment greater than 15%.

Therefore, it appears to be wiser to directly replicate the equity variable in this approach because estimation error of the capital remains homogeneous to the replication errors.

These two methods are illustrated in section 6 of this article.

6. APPLICATIONS

6.1 Presentation of tested products

We tested the alternative method presented herein on a French saving portfolio with low average guaranteed minimum rates (the majority of contracts are 0% guaranteed rates, and with the remaining being 4% guaranteed rates). The projection tool that was used enables the taking into account of profit sharing mechanisms, target crediting rate and dynamic lapses behaviors of policy holders.

The asset allocation rule is such that the initial asset allocation is maintained over the duration of the projection. The initial allocation is the following:

Asset	Allocation
Stock	10%
Bonds	70%
Cash	20%

A specific rule of profits sharing between the insurer and the clients is modeled. The credited rate that is paid to clients is calculated as a function of the risk-free rates and the

[1] This condition is satisfied the 12/31/2008 by most of life insurance companies on the French market.

performance of the CAC index, and of the profit sharing rules. A dynamic lapse function has also been defined based on the aforementioned crediting rate and a market based target crediting rate : if the crediting rate is too low, a proportion (calculated within the model) of policy holders will redeem their contracts.

The only two risks that are considered are the stock risks and the interest rate risks. The economic scenario tables are calibrated as of the 12/31/2008.

The method gives similar results for different hypothesis or different products. The change of a hypothesis (asset allocation, economic information, ALM rules…) implies a new calibration of the Replicating Portfolio.

6.2 Results

We have carried out a NS projection based on 15'000 real world simulations so as to create a reference economic capital. For each primary simulation, the equity at $t = 1$ was estimated with recalibrated and regenerated risk-neutral simulations. The study of the quality of fit between RP and equity distribution provided by the model enables to validate the calculations at each step of the process.

The following table lists the results achieved for the studied product:

$\boldsymbol{Eq_0}$	$\boldsymbol{P(0,1)q_{0.5\%}\widehat{(Eq_1})}$	$\boldsymbol{C_{NS}}$
25.90	0.58	25.32

Table 2 : results of the complete NS projection

In this section we shall provide a detailed description of the use of the alternative method with the studied product, and then we shall analyze the results of this method in the context of the replication of the liabilities' best estimate. Finally we shall present the calculations that stem from a “replication of the cash flows' present value" approach at the initial date.

6.2.1 Calibration on shareholder’s equity

A preliminary study enabled us to determine the equity values associated with the most adverse simulations and to extract the interest rates and stock risk factors for each simulation by using the real-world scenarios' table. We considered $N = 150$ extreme equity values for the analysis of convexity and for the determination of the RP's asset mix (steps 0, 1 and 3).

The study of the convexity of equity at $t = 1$ led us to the following parametric form structure:

$$Eq_{param} = A_{0,0} + A_{1,0} \cdot \varepsilon_S + A_{2,0} \cdot \varepsilon_S^2 + A_{3,0} \cdot \varepsilon_S^3 + A_{0,1} \cdot \varepsilon_{ZC} + A_{0,2} \cdot \varepsilon_{ZC}^2 + A_{1,1} \cdot \varepsilon_S \cdot \varepsilon_{ZC}.$$

The results obtained for the economic capital and marginal equity with the parametric form calibrated by 150 scenarios are as follows:

	Global SCR	**Stock marginal Eq**	**Interest rate marginal Eq**
NS Value	25.32	5.84	23.21
Parametric form value	25.10	5.84	23.32
Relative difference	-0.87%	0.06%	0.49%

Table 3: comparison of NS and parametric form results

The assessment of global and marginal capitals with the parametric approach yields highly satisfactory results on the product studied here.

We established 5 sub-RPs that enable to replicate the following terms: ε_S , ε_S^2, ε_S^3, ε_{ZC} and ε_{ZC}^2.

As mentioned above, the $\varepsilon_S \cdot \varepsilon_{ZC}$ term is not specifically replicated; the crossed effects are taken in by the stock derivatives of the consolidated RP.

The candidate assets used in the sub-RP comply with the mapping presented in section 4.5.3.

Below is a detailed description of the assets and parameters of the portfolio $RP^{(2,0)}$ that enables the replication of the term ε_S^2 :

	Call 1	**Call 2**	**Put 1**	**Put 2**
Exercise date	4	3	3	4
Strike	101.4	133.2	75.0	101.7

Table 4: characteristics of the portfolio $\boldsymbol{RP^{(2,0)}}$

We also provide a description of the characteristics of the $RP^{(0,2)}$ portfolio that replicates the term ε_{ZC}^2:

	Caplet 1	Caplet 2	Floorlet 1	Floorlet 2
Strike rate	2.50%	3.22%	6.08%	4.52%
Exercise date	3	3	3	3
Maturity of the underlying rate	1	1	1	1

Table 5: characteristics of the portfolio $\boldsymbol{RP}^{(\mathbf{0,2})}$

Finally the table given hereunder lists the properties of the $RP^{(3,0)}$ portfolio that replicates the term ε_S^3 :

	Call 3	Call 4	Put 3	Put 4
Exercise date	2	2	2	2
Strike	114.7	165.8	75	75.1

Table 6: characteristics of the portfolio $\boldsymbol{RP}^{(\mathbf{3,0})}$

The total number of candidate assets that make up the aggregated RP is of 14. It should be mentioned that the assets used show no degree of complexity (furthermore, the derivatives considered in our portfolio are "plain vanilla").

We then established the weight of the portfolio's 14 assets by minimizing the criterion presented in step 3 on the basis of 150 extreme equity values calculated in steps 0 and 1.

The RP's market value was then projected over the 15'000 real-world simulations and we studied the adjustment of the “NS equity” variable vs. the “RP price at $t = 1$” variable.

We gave ourselves a 1% error limit in the replication of the initial value of equity. The resolution of the optimization program induced a saturation of this constraint. In other words, one obtains the following equality:

$$\frac{\left|Eq_0 - \sum_{k=1}^{M} w_k \, A_k\,(0)\right|}{Eq_0} = 1\%.$$

The following table presents the values of economic capital by NS and RP:

SCR NS	SCR RP	Relative deviation
25.32	25.34	0.05%

Table 7 : comparison of economic capitals NS and RP

It reveals that the alternative method yields very good results. It should also be noted that the 0.5% deviation between NS and RP capitals is largely inferior to the NS/parametric form deviation (which is of 0.87% in absolute value). Even if the triangle inequality provides an upper bound of the error, this example shows that the global error does not stem from an addition of the errors committed at each step, because the asset mix is determined in one final phase on a consolidated portfolio (the estimated weight during sub-replication is not maintained).

Besides, a QQ-plot of Nested Simulation values of equity versus Replicating Portfolio market values and a Kolmogorov-Smirnov test (with a P-value close to 100%) allow us to validate the adequacy of both distributions.

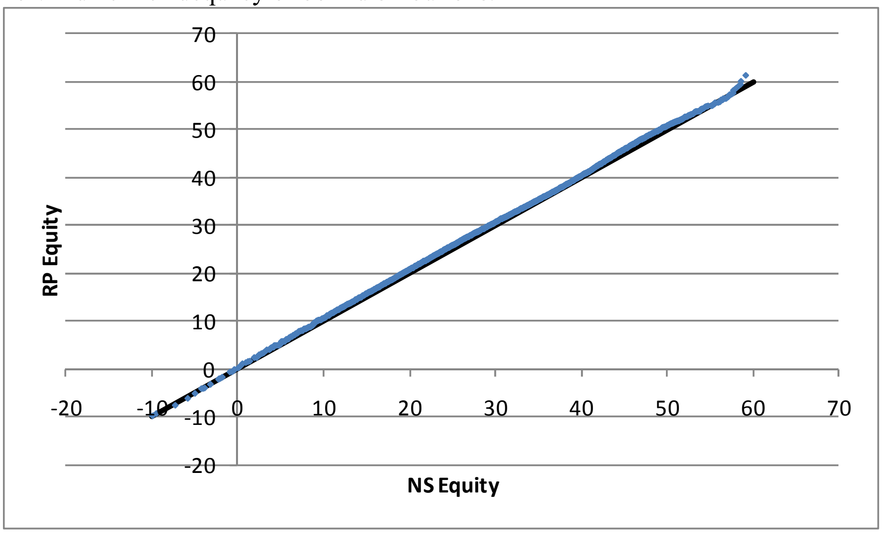

Figure 3 : Equity distribution " NS methodology vs Replicating Portfolios"

6.2.2 Calibration on the Best Estimate of Liabilities

As we explained in part 4, the best estimate of the liabilities can be replicated instead of that of the equity. In this case, the RP may be built in a manner that is similar to the method described above. One has only to work on the extreme values of the liabilities' Best Estimate in steps 0, 1 and 3.

We achieved a highly satisfactory replication of the best estimate of liabilities by retaining the same assets as those used for calibration on equity[1]. The deviation inherent to

[1] Since the liabilities' economic value is the difference between the market value of the company's asset and the economic equity, the set of candidate assets remains unchanged. The only item that differs is the weighting of the assets.

the percentile scenario (i.e. the scenario that leads to the $[0{,}5\%.P]^{\text{th}}$ worst distribution value for equity among the 15'000 real-world scenarios) between the value of the liabilities calculated according to NS vs. RP is approximately of 1%. This error can be hardly been reduced, whatever the portfolio replicated.

The distribution of equity was then calculated as the difference between the market value of the company's asset simulated over one year and its RP value.

This application gives rise to a deviation between the NS capital and the RP capital of more than 15%, as shown in the following table:

	NS	**Replicating Portfolio**	**Deviation**
Value of liabilities in the percentile scenario	1 045.6	1 056.5	1.05%
Economic capital	25.32	29.52	16.6%

Table 8: comparison of results of NS and liability-adjusted RP

One observes a very high deviation between the two amounts of capital. As mentioned previously, this stems from the fact that a very slight error of replication at the level of the liabilities may lead to very unsound economic capital assessments (as the error is multiplied by an amplification coefficient). However, the same set of candidate assets (with identical parameters) leads to a 0.05% deviation in the context of a direct replication of equity.

It therefore seems highly preferable to try to replicate equity rather than liabilities.

6.2.3 Calibration of the cash flows' present value

In this section we present the results of an application of the "replication of the cash flows' present value" technique. It relies on the method presented in section 3.2.3 and is based on present value of future profit rather than on liability cash flows. We have retained the best candidate assets obtained with the alternative methods. The asset mix of the replicating portfolio was calibrated by regression of the PVCF results at $t = 0$ based on a set of 1000 risk-neutral simulations of present values of the candidate assets' cash. For added robustness in the calibration process, we added 8 shocks replication constraints (see criterion P3 explained in section 3.2.2).

The regression therefore required the simulation of 1000 risk-neutral scenarios (the complexity is similar to a MCEV calculation) and the knowledge of 8 shocked MCEV values at $t = 0$.

The RP value was then projected at $t = 1$ over the 15'000 real-world simulations in order to determine the economic capital.

The following table compares the NS capital with the capital resulting from the assessment of the RP thus calibrated:

SCR NS	**SCR RP**	**Relative deviation**
25.32	24.36	3.78%

Table 9: comparison of capitals according to NS and RP calibrated on PVCF

One observes that this calibration method leads to less satisfactory results compared with the alternative method presented herein. This observation has been noticed for different products with different design.

It should be noted that the robustness of the PVCF replication method relies essentially in the addition of shock replication constraints.

CONCLUSION

In this article we have demonstrated that the use of a Replicating Portfolios method is a powerful tool in the determination of the distribution of equity over a one-year period. Indeed, this approach enables the rapid construction of large empirical distributions for the assessment of capital needs.

However, as we explained, when using standard calibration techniques one has to pay attention on the following points: the make-up of the portfolio and the parameters of its assets, the selection of the optimization program, the type of the variable that is to be replicated...

Following a theoretical analysis of these various points, we demonstrated by means of an example that even a satisfactory replication of the best estimate of liabilities may lead to weak assessments of the economic capital. We therefore deem it essential to Endeavour to replicate directly the equity. We also demonstrated that a method that relies on the replication of present value of future profits could give rise to a loss of accuracy when assessing the economic capital. It should be noted that the addition of shock replication constraints is crucial in this context and provides more robustness to the portfolio calibration process.

Our alternative method enables, by means of an analysis of the convexity of the "equity" function, to associate each term of a parametric form with a sub-replicating portfolio, the parameters of which can be set in an automatic and optimal fashion. However, the use of the alternative method requires that the set of shocked equity values is known and may lead to longer calibration times than the commonly used techniques. We tested this method on various life-insurance portfolios and the results systematically yielded a robust estimation of the Solvency II economic capital.

We developed several versions of the calibration method presented herein that rely on “hybrid realizations” (i.e. that mix real-world and risk-neutral universes within a simulation) of random variables rather than the knowledge of conditional expectations. These techniques imply much shorter calculation times. These various studies will be the subject of forthcoming publications.

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