SEASONALITY MODELLING FOR CATASTROPHE BOND PRICING

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Abstract:

During the last decades, a new category of assets whose return is linked to insurance claims have appeared. Those assets, called catastrophe bonds, are primarily designed by insurers and reinsurers to transfer their risks to other categories of investors, looking for diversification. This paper proposes a method to price such bonds, when the claims arrival process is under the influence of a stochastic seasonal effect. The arrival process is modeled by a Poisson Process whose intensity is the sum of an Ornstein Uhlenbeck process and of one periodic function. The size of claims is assumed to be a positive random variable, independent of the intensity process. In this paper, we show that the expected number of claims can be inferred from the probability generating function and propose a pricing method of the fair coupon based on the Fourier Transform. To illustrate the tractability of our model, we price insurance bonds on claims resulting from tornadoes in the US.

Keywords: catastrophe bonds, doubly stochastic processes, Fast Fourier Transform.

1. INTRODUCTION.

During the last two decades, we have attended to the emergence of a new category of assets, primarily developed to hedge the costs of insuring natural catastrophes. In this context, catastrophe insurance derivatives have been introduced at the Chicago Board of Trade in the early nineties. The value of these securities is directly related to indexes that account the total insurance losses due to natural catastrophes in US, by regions. Reinsurers have also started to propose a wide range of insurance bonds, based upon the mechanism of securitization. Those products offer two advantages. Firstly, they transfer a part of insurance risks from the reinsurers to other potential investors and allow then to increase the reinsurers' volume of transactions. Secondly, insurance derivatives are efficient tools of diversification for institutional investors, that are not exposed in their core business to

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catastrophe risks. Indeed, the securities linked to insurance events are not at all correlated with financial markets. The development and pricing of insurance bonds is closely related to the one of credit derivatives and of securitized credits. The interested reader can refer to the paper of Schwartz and Torous (1989) for an introduction to valuation of Mortgage-Backed Securities.

However, the valuation of catastrophe or insurance derivatives is obviously more complex compared to the pricing of purely financial securities. The first problematic element is the incompleteness of the insurance linked securities, given that the underlying risks are not tradable, by nature. This point has been underlined by many authors, see for e.g. Muermann (2001), Charpentier (2007). To summarize, the incompleteness entails that there exist more than one risk neutral measure, and that the price is not unique. A second issue related to pricing is the complexity of the aggregate losses process, that could hardly be modeled by standard financial tools. The first attempts of pricing were done by Cummins and Geman (1994), Geman and Yor (1997). They model the underlying catastrophe indexes by a geometric Brownian with jumps. Aase (1999,2001), Christensen and Schmidli (2000) proposed to model the aggregated losses as a compound Poisson process with stochastic size of claims. In a recent work, Biagini et al. (2008) studied the valuation of catastrophe derivatives on a loss index but with a reestimation of the total aggregated claims. In these papers, the intensity of the Poisson process determining the number of claims, is either constant or either a deterministic function of time. Similar approaches are studied by Roustan (2003) in his PhD thesis. Jang (2000), Dassios and Jang (2003) improved the modeling of the aggregated losses by assuming that the claims arrival process is driven by a Poisson process whose the intensity is a stochastic shot noise process. The pricing of the insurance derivatives is done under the Esscher measure. For a survey on the Esscher measure, see e.g. Miyahara (2004). We can also draw a parallel with the valuation of securitized products related to life policies. They also require to understand and to model the surrender behaviors of insured. We refer to the works of Albizzati and Geman (1994) or Le Courtois and Nakagawa (2012) for valuation of surrender options. The paper of Milhaud et al. (2011) explores the surrender behavior with a CART model.

The contribution of this paper is to propose a method to price an insurance bond paying multiple coupons and related to a seasonal arrival process of claims. The interest for modeling the seasonality is particularly obvious for claims such tornadoes, hurricanes, storms, flooding or even car accidents which are more frequent during certain periods of the year. For many insurance securities, it is then crucial to integrate this trend in the pricing. In our approach, the claims arrival process is modeled by a doubly stochastic process, whose intensity is the sum of a deterministic seasonal function and of a mean reverting stochastic process. We next rely on Fast Fourier Transform to price insurance bonds. The method developed in this article is quite general and may be applied to any seasonal insurance claims. To illustrate the tractability of our model, we apply the methodology to price a cat bond linked to claims caused by tornadoes having the US between 1974 and 1998.

The first section of this paper details the claims arrival process and provides a recursion to compute the expected number of claims. In section 3, the aggregated claims process and the insurance bond are defined. We also present the general formula to value the coupon rate of such bonds. Section 4 reveals how the Fast Fourier Transform can help us to solve to price bonds. The last section is devoted to a numerical application that underlines the feasibility of our approach.

2. THE CLAIMS ARRIVAL PROCESS.

The starting point of this research is the observation that many natural phenomenons are seasonal. For example, we retrieved on Sheldus[1] (Spatial Hazard Events and Losses Database for the United States) the monthly numbers of tornadoes having hit the US between 1961 and 2008. Those figures are illustrated by the graph 0 which clearly reveals that more tornadoes are observed during the second term than during the remainder of the year. Modeling the number of claims by a Poisson process having a constant intensity may not capture this trend.

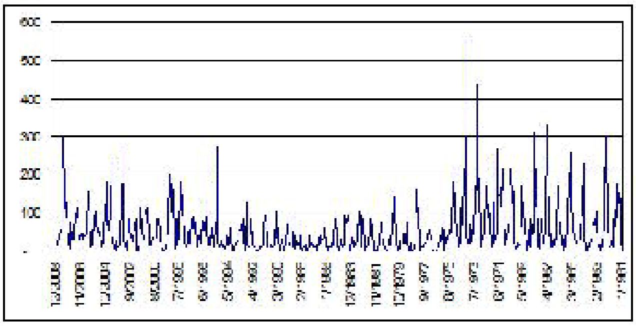

Figure 1: Number of Tornadoes in the US.

[1]http://webra.cas.sc.edu/hvriapps/sheldus_setup/sheldus_login.aspx

So as to capture the seasonality of natural phenomenons, we have assumed in this paper that the number of claims observed till time t, is a Poisson process, noted N_t, having a stochastic intensity, and whose the mean is a periodic function. This class of processes is called doubly stochastic and has already been widely used to model the process of credit events. It also seems well adapted to model the arrivals of claims. The process N_t is defined on a filtration $\mathcal{F}_t$, in a probability space Ω coupled to a probability measure, noted Q. This measure is assumed to be the risk neutral measure, used for pricing purposes (the relation between the modeling under P , the real measure, and Q is developed in appendix A). A common practice adopted by actuaries is to assume that the parameters ruling the claims process are identical under P and Q. We will adopt this assumption in the example of section 5. The intensity of N_t is a stochastic process, that we note λ_t, defined on a filtration $\mathcal{H}_t$ such that conditionally on $\mathcal{H}_t \vee \mathcal{F}_0$, the process N_t is a Poisson process for which, the probability of observing k jumps is given by the formula:

$$P(N_t = k|\mathcal{H}_t \vee \mathcal{F}_0) = \frac{\left(\int_0^t \lambda_u du\right)^k}{k!} e^{-\int_0^t \lambda_u du} . \tag{1}$$

For more details on doubly stochastic processes, we refer the interested reader to Bremaud (1981, chapter 2) and Bielecki & Rutkowski (2004 ,chapter 6). The frequency of many claims presents both stochasticity and seasonality. This is obviously the case for claims related to natural calamities, such storms or flooding but this also the case for car accidents whose frequency climbs during the winter period. So as to capture this double characteristic, stochasticity and seasonality, in the pricing of insurance bonds, the intensity of our Poisson Process is modeled as the sum of a cyclical deterministic function $\lambda(t)$, and of a stochastic process λ_t^{OU}:

$$\lambda_t = \lambda(t) + \lambda_t^{OU} . \tag{2}$$

The deterministic cyclical function is defined by three real constant parameters $\delta, \beta, \gamma \in \mathbb{R}$:

$$\lambda(t) = \delta + \beta \cos\big((t+\gamma)2\pi\big) . \tag{3}$$

Note that replacing λ_t by any other integrable function would not affect the developments done in the remainder of this work. Lu and Garrido (2005) used a similar approach with a beta function whose parameters varies according to a switching process. While the stochastic component of the intensity, λ_t^{OU}, is an Ornstein Uhlenbeck process, whose the speed and level of mean reversion and the volatility are real constants, respectively denoted $a, b, \sigma \in \mathbb{R}$. The dynamic of λ_t^{OU} is ruled by the following stochastic differential equation:

$$d\lambda_t^{OU} = a(b - \lambda_t^{OU})dt + \sigma dW_t \quad , \tag{4}$$

where W_t is a Brownian motion defined on the filtration $\mathcal{H}_t$. We have chosen to work with an Ornstein Uhlenbeck process for his analytical tractability. The figure 1 presents an example of two trajectories followed by the intensity process. The distribution of λ_t is detailed in the next proposition.

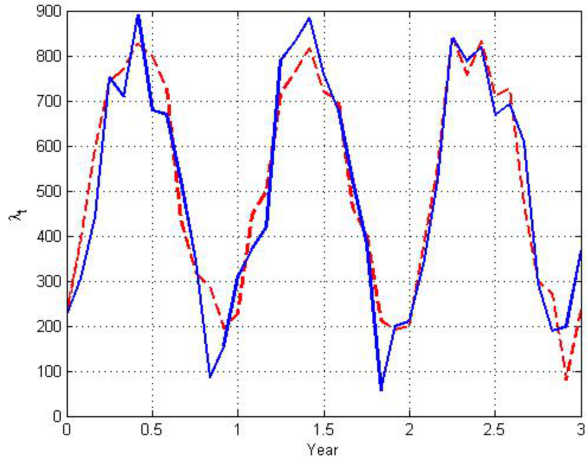

Figure 2: Example of path for λ_t.

Proposition 2.1 *In our model, the process λ_t is a Gaussian random variable conditionally on $\mathcal{H}_s$, whose the average, $\mu^{\lambda}(s,t)$, and variance $\left(\sigma^{\lambda}(s,t)\right)^2$ are given by the following expressions:*

$$\mu^{\lambda}(s,t) = \lambda(t) + e^{-a(t-s)}\lambda_s^{OU} + b(1 - e^{-a(t-s)}) \tag{5}$$

$$\left(\sigma^{\lambda}(s,t)\right)^2 = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right) \tag{6}$$

Proof. We just sketch the proof because this result is rather standard and we refer the reader to Musiela & Rutkowski 1997, chapter 12 p. 289, for details. The first step consists of differentiating the process $Z_t = e^{at}(b - \lambda_t^{OU})$ to show that

$$Z_t = Z_s - \int_s^t e^{au}\sigma dW_u, \tag{7}$$

and as $\lambda_t^{OU} = b - e^{-at}Z_t$, we infer from eq. (7), that

$$\lambda_t^{OU} = e^{-a(t-s)}\lambda_s^{OU} + b(1 - e^{-a(t-s)}) + \int_s^t e^{-a(t-u)}\sigma dW_u \ . \tag{8}$$

The results of the proposition directly follow from this last relation.

The intensity process, λ_t solution of eq. (4) being a Gaussian random variable, the probability of observing a negative value for this process is not null. However, when the annual average level δ of $\lambda(t)$ and level of mean reversion b are sufficiently high compared to the volatility σ, this probability should be infinitesimal, and λ_t may be used as intensity for N_t. Working with the CIR model would be an alternative to avoid this issue but in this case, we lose the analytical tractability of the Ornstein Uhlenbeck process. We now present two propositions that allow us later to determine the probability generating function of N_t.

Proposition 2.2 *The integral of* λ_u *from* t_1 *to* t_2, *is a Gaussian random variable conditionally on* $\mathcal{H}_s$, *whose the average,* $\mu^{\int \lambda du}(s,t_1,t_2)$, *and variance* $\left(\sigma^{\int \lambda du}(s,t_1,t_2)\right)^2$ *are given by the following expressions:*

$$\mu^{\int \lambda du}(s,t_1,t_2) = \delta(t_2-t_1)+\frac{1}{2\pi}\beta\sin\left(2\pi(t_2+\gamma)\right)-\frac{1}{2\pi}\beta\sin\left(2\pi(t_1+\gamma)\right)$$
$$+\lambda_s^{OU}e^{-a(t_1-s)}B(t_1,t_2)+b\left((t_2-t_1)-e^{-a(t_1-s)}B(t_1,t_2)\right) \tag{9}$$

$$\left(\sigma^{\int \lambda du}(s,t_1,t_2)\right)^2 = \frac{\sigma^2}{2a}B(t_1,t_2)^2\left(1-e^{-2a(t_1-s)}\right)+$$
$$\frac{\sigma^2}{a^2}\left((t_2-t_1)-B(t_1,t_2)-\frac{1}{2}aB(t_1,t_2)^2\right) \tag{10}$$

where the function $B(t_1,t_2)$ *is defined as follows:*

$$B(t_1,t_2)=\frac{1}{a}\left(1-e^{-a(t_2-t_1)}\right)$$

Proof. The integral of λ_u is the sum of the integrals of $\lambda(u)$ and of λ_u^{OU}. The integral of $\lambda(u)$ from t_1 to t_2 is worth:

$$\int_{t_1}^{t_2}\lambda(u)du = \delta(t_2-t_1)+\frac{1}{2\pi}\beta\sin\left(2\pi(t_2+\gamma)\right)-\frac{1}{2\pi}\beta\sin\left(2\pi(t_1+\gamma)\right), \tag{11}$$

whereas the integral of the process λ_u^{OU} is obtained by integrating the eq. (8) from t_1 to t_2:

$$\int_{t_1}^{t_2}\lambda_u^{OU}du = \lambda_s^{OU}e^{-a(t_1-s)}B(t_1,t_2)+b\left((t_2-t_1)-e^{-a(t_1-s)}B(t_1,t_2)\right)$$
$$+\int_{t_1}^{t_2}\frac{\sigma}{a}\left(e^{-at_1}-e^{-at_2}\right)e^{au}dW_u+\int_{t_1}^{t_2}\sigma B(u,t_2)dW_u\,. \tag{12}$$

The results of the proposition directly follow from this last relation.

Conditionally on $\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1} \supset \mathcal{F}_s$, the process $N_{t_2} - N_{t_1}$ is a Poisson process. This property allows us to deduce that the probability of observing k jumps in a certain interval of time is equal to the following expectation where I is an indicator variable:

$$\begin{aligned} P(N_{t_2} - N_{t_1} = k|\mathcal{F}_s) &= \mathbb{E}\left(I_{N_{t_2}-N_{t_1}=k} \,|\mathcal{F}_s \right) \\ &= \mathbb{E}\left(\mathbb{E}\left(I_{N_{t_2}-N_{t_1}=k} \,|\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1} \right) |\mathcal{F}_s \right) \\ &= \mathbb{E}\left(P\left(N_{t_2} - N_{t_1} = k|\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1} \right) |\mathcal{F}_s \right) \\ &= \mathbb{E}\left(\frac{\left(\int_{t_1}^{t_2} \lambda_u du \right)^k}{k!} e^{-\int_{t_1}^{t_2} \lambda_u du} |\mathcal{F}_s \right). \end{aligned} \tag{13}$$

Except for $k = 0$, no analytical expression exists for this last expectation. However, it will be shown in the remainder of this section that the probabilities of observing $k > 0$ jumps, can be computed by means of an iterative procedure based upon the probability generating function (pgf) of N_t , as defined in the next proposition.

Proposition 2.3 *The pgf of* N_t *is given by the following expectation:*

$$\begin{aligned} pgf(x,s,t_1,t_2) &= \mathbb{E}\left(x^{N_{t_2}-N_{t_1}} \,|\, \mathcal{F}_s \right) = \mathbb{E}\left(e^{\int_{t_1}^{t_2} \lambda_u du (x-1)} \,|\, \mathcal{F}_s \right) \\ &= \exp\left((x-1)\mu^{\int \lambda du}(s,t_1,t_2) + \frac{1}{2}(x-1)^2 \left(\sigma^{\int \lambda du}(s,t_1,t_2) \right)^2 \right) \end{aligned} \tag{14}$$

Proof. Given that $\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1} \supset \mathcal{F}_s$, we can rewrite the probability generating function as follows:

$$\begin{aligned} pgf(x,s,t_1,t_2) &= \mathbb{E}\left(x^{N_{t_2}-N_{t_1}} \,|\, \mathcal{F}_s \right) \\ &= \mathbb{E}\left(\mathbb{E}\left(x^{N_{t_2}-N_{t_1}} \,|\, \mathcal{H}_{t_2} \vee \mathcal{F}_{t_1} \right) |\, \mathcal{F}_s \right), \end{aligned}$$

and as conditionally on $\mathcal{H}_{t_2} \vee \mathcal{F}_{t_1}$, the jump process is Poisson whose the mgf is known and given by

$$\mathbb{E}\left(e^{y\left(N_{t_2}-N_{t_1}\right)} \,|\, \mathcal{H}_{t_2} \vee \mathcal{F}_{t_1} \right) = e^{\int_{t_1}^{t_2} \lambda_u du (e^y - 1)}$$

resulting in

$$pgf(x,s,t_1,t_2) = \mathbb{E}\left(\mathbb{E}\left(x^{N_{t_2}-N_{t_1}} \mid \mathcal{H}_{t_2} \vee \mathcal{F}_{t_1}\right) \mid \mathcal{F}_s\right)$$

$$= \mathbb{E}\left(\mathbb{E}\left(e^{ln(x)\left(N_{t_2}-N_{t_1}\right)} \mid \mathcal{H}_{t_2} \vee \mathcal{F}_{t_1}\right) \mid \mathcal{F}_s\right)$$

$$= \mathbb{E}\left(e^{\int_{t_1}^{t_2} \lambda_u du\,(x-1)} \mid \mathcal{F}_s\right). \tag{15}$$

The integral of the intensity being Gaussian according to proposition 2.2, the expectation in eq. (15) is the one of a lognormal variable and is well equal to eq. (14).

The pgf is a powerful tool that give us the possibility to infer by a recursive method, detailed in the next proposition, the probabilities of jumps of $N_{t_2} - N_{t_1}$, conditionally on the filtration $\mathcal{F}_s$,at time s .

Proposition 2.4 *The probability not to observe any jumps in the interval of time* $[t_1,t_2]$ *, conditionally on* $\mathcal{F}_s$ *is equal to*

$$P(N_{t_2} - N_{t_1} = 0|\mathcal{F}_s) = pgf(x,s,t_1,t_2)|_{x=0}$$

$$= \exp\left(-\mu^{\int \lambda du}(s,t_1,t_2) + \frac{1}{2}\left(\sigma^{\int \lambda du}(s,t_1,t_2)\right)^2\right). \tag{16}$$

The probability that the process N_t *exhibits exactly one jump in the interval of time* $[t_1,t_2]$ *is equal to:*

$$P(N_{t_2} - N_{t_1} = 1|\mathcal{F}_s) = \frac{\partial}{\partial x} pgf(x,s,t_1,t_2)\Big|_{\{x=0\}}$$

$$= P(N_{t_2} - N_{t_1} = 0|\mathcal{F}_s)\left(\mu^{\int \lambda du}(s,t_1,t_2) - \left(\sigma^{\int \lambda du}(s,t_1,t_2)\right)^2\right). \tag{17}$$

The probability of observing more k *jumps can be computed iteratively as follows:*

$$P(N_{t_2} - N_{t_1} = k|\mathcal{F}_s) = \frac{1}{k!}\frac{\partial}{\partial x^k} pgf(x,s,t_1,t_2)\Big|_{\{x=0\}} \tag{18}$$

where

$$\frac{\partial^k}{\partial x^k} pgf(x,s,t_1,t_2)\Big|_{\{x=0\}} =$$

$$\left(\mu^{\int\lambda du}(s,t_1,t_2)-\left(\sigma^{\int\lambda du}(s,t_1,t_2)\right)^2\right)\frac{\partial^{k-1}}{\partial x^{k-1}}pgf(x,s,t_1,t_2)\Big|_{\{x=0\}}$$

$$+(k-1)\left(\sigma^{\int\lambda du}(s,t_1,t_2)\right)^2\frac{\partial^{k-2}}{\partial x^{k-2}}pgf(x,s,t_1,t_2)\Big|_{\{x=0\}} \quad (19)$$

Proof. The proof of this proposition directly results from the exponential form of the probability generating function.

The proposition 2.4 provides us with an important tool in order to calibrate our model parameters ($\delta,\beta,\gamma,a,b,\sigma$) to real data. The calibration can be done by maximizing the likelihood of observed numbers of claims, on several seasons.

3. THE SIZE OF CLAIMS AND THE PRICING OF BONDS.

The risk faced by an insurance bondholder is inherent to his exposure to accumulated insured property losses. This process of accumulated losses, which is denoted by X_t in the sequel of this work, depends both upon the frequency of claims N_t and on the magnitude of claims. The size of the j^{th} claim, occurring at time t_j, is modeled by a positive random variable, Y_j, defined on the filtration $\mathcal{F}_{t_j}$, but assumed to be independent from previous claims and from the frequency. There is no other constraint on the choice of Y_j. As for the claims arrival process N_t, we directly work with the distribution of Y_t under the risk neutral measure, Q (again we refer the interested reader to appendix A for details about the relation between the modeling under P, the real measure, and Q). The process of aggregated losses, is defined by the following expression:

$$X_t = \sum_{t_j<t} Y_j$$
$$= \sum_{j=1}^{N_t} Y_j$$

We now describe the characteristics of an insurance bond and develop the method to price such kind of assets. The insurance bond periodically pays a coupon equal to a constant percentage of the nominal reduced by the amount of aggregated losses, exceeding a certain trigger level. At maturity, what is left of the nominal value is repaid. In order to compensate for this eventual loss of nominal, the coupon rate always exceeds the risk free rate. In case a few claims occur, the bondholder is then rewarded at a higher rate than the one obtained by investing in risk free assets with same maturities. On the contrary, in case of catastrophic losses, the nominal of the bond can fall to zero and the payment of coupons can be interrupted. To understand how the spread of this bond is priced, we need to introduce some additional mathematical notations.

Let us use the notation BN, the initial nominal of the bond. The level above which the excess of aggregated losses is deduced from the nominal, is noted K_1. If the total insured losses reach the amount of $K_2 = K_1 + BN$, before maturity, the bond stops to deliver coupons and the nominal is depleted. The bond, issued at time t_0, pays n coupons, at regular intervals of time, Δt, ranging from t_1 to t_n. The coupon rate is the sum of the constant risk free rate of maturity t_n, and of a spread, that are respectively noted r and sp. The coupons paid at times $t_{i=1\ldots n}$ are noted $cp(t_i)$ and defined as follows:

$$cp(t_i) = (r+sp)\Delta t \underbrace{\left[\left(K_2 - K_1\right) I_{X_{t_i} \in [0,K_1]} + \left(K_2 - X_{t_i}\right) I_{X_{t_i} \in (K_1, K_2]} \right]}_{BN_{t_i}}. \tag{20}$$

The term between brackets is the (stochastic) nominal of bond at time t_i and is written BN_{t_i} in the sequel of our developments. Note that BN_{t_0} is worth BN. Based upon the principle of absence of arbitrage, the spread of the insurance bond is chosen such that the expectations of future discounted spreads and of future discounted cutbacks of nominal are equal, under the risk neutral pricing Q. The expectations of future discounted spreads and reductions of nominal are respectively named the ``spreads leg" and the ``claims leg" . They are defined by the following expressions:

$$Spreadsleg(t_0) = sp\Delta t \sum_{t_i = t_1}^{t_n} e^{-r(t_i - t_0)} \mathbb{E}\left(BN_{t_i} | \mathcal{F}_{t_0} \right), \tag{21}$$

$$Claimsleg(t_0) = \sum_{t_i = t_1}^{t_n} e^{-r(t_i - t_0)} \mathbb{E}\left(BN_{t_{i-1}} - BN_{t_i} | \mathcal{F}_{t_0} \right). \tag{22}$$

By equating equation (21) and equation (22), we infer the following fair spread rate that should be added to the risk free rate, at the issuance of the insurance bond:

$$sp = \frac{\sum_{t_i = t_1}^{t_n} e^{-r(t_i - t_0)} \mathbb{E}\left(BN_{t_{i-1}} - BN_{t_i} | \mathcal{F}_{t_0} \right)}{\Delta t \sum_{t_i = t_1}^{t_n} e^{-r(t_i - t_0)} \mathbb{E}\left(BN_{t_i} | \mathcal{F}_{t_0} \right)} \tag{23}$$

Despite the apparent simplicity of this last expression, the expected future nominals are not calculable by a closed form equation and we have to rely on numerical methods to appraise them. Among the numerical tools available, we have chosen the Fourier transform.

But before detailling this numerical method, we draw the attention of the reader on the similarities between the formula (3) and the pricing of a credit default swap (CDS). A credit default swap is an insurance protecting the owner of a corporate bond, of principal N, against the default of the bond issuer. In exchange of regular payments (named the premium leg that is equivalent to our spread leg), the buyer of the CDS receives the part of the bond principal which is not repaid in case of bankruptcy of the bond issuer. The payment done if default occurs, is called the default leg and is identical to our claims leg. The premium paid is usually expressed as a percentage of the bond principal. We assume that premiums are paid at regular intervals of time, Δt, ranging from t_1 to t_n and are expressed as a coupon p of the nominal. The time of default of the bond issuer is noted τ. Then the premium leg is equal to

$$Premiumleg(t_0) = p\Delta t \sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathbb{E}\left(N I_{t_i<\tau} | \mathcal{F}_{t_0} \right) \tag{24}$$

If the bond issuer goes to bankrupcty, the CDS pays the difference between the principal and the random recovery rate noted R_τ, the default leg is then

$$Defaultleg(t_0) = \sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathbb{E}\left(\left(N - R_\tau \right) I_{t_{i-1}<\tau<t_i} | \mathcal{F}_{t_0} \right). \tag{25}$$

By equating equation (24) and equation (25), we infer the CDS premium rate:

$$p = \frac{\sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathbb{E}\left(\left(N - R_\tau \right) I_{t_{i-1}<\tau<t_i} | \mathcal{F}_{t_0} \right)}{\Delta t \sum_{t_i=t_1}^{t_n} e^{-r(t_i-t_0)} \mathbb{E}\left(N I_{t_i<\tau} | \mathcal{F}_{t_0} \right)}. \tag{26}$$

A comparison of equations (23) and (26) reveals the similarities between the CDS and Cat bond pricing.

4. PRICING BY FOURIER TRANSFORM.

As explained in the previous paragraph, the pricing of an insurance bond requires the valuation of the expected future value of nominal. According to eq. (20), this expectation may be split into two components,

$$\mathbb{E}\left(BN_{t_i} | \mathcal{F}_{t_0} \right) = \mathbb{E}\left[BN\, I_{X_{t_i} \in [0,K_1]} | \mathcal{F}_{t_0} \right] + \mathbb{E}\left[\left(K_2 - X_{t_i} \right) I_{X_{t_i} \in (K_1,K_2]} | \mathcal{F}_{t_0} \right]. \tag{27}$$

As done by Carr and Madan (1999), each component can be reformulated in terms of their Fourier transforms. The two next propositions deal with this point.

Proposition 4.1 *The following expectation may be rewritten as follows:*

$$\mathbb{E}\left[BN\, I_{X_{t_i} \in [0,K_1]} | \mathcal{F}_{t_0} \right] =$$

$$\frac{BN}{\pi} \int_0^{+\infty} \varphi^1(u) e^{-iuX_{t_0}} \sum_{k=0}^{+\infty} P(N_{t_i} - N_{t_0} = k) \left(\mathbb{E}\left(e^{-iuY} \right) \right)^k du \qquad (28)$$

where $\varphi^1(u)$ *is the Fourier transform of the function* $I_{x \in [0,K_1]}$, $x \in \mathbb{R}^+$,

$$\varphi^1(u) = \frac{1}{iu}\left(e^{iuK_1} - 1 \right). \qquad (29)$$

The probabilities $P(N_{t_i} - N_{t_0} = k)$ *can be retrieved from proposition 2.4 whereas the expectation* $\mathbb{E}\left(e^{-iuY}\right)$ *is the Laplace transform of the claim size* Y .

Proof. Let us denote by $q_{X_{ti}|\mathcal{F}_{t0}}(x)$ the density of the aggregated losses process X_{ti}, conditionally on the filtration $\mathcal{F}_{t0}$. We can rewrite the expectation i (36) as follows:

$$\mathbb{E}\left[I_{X_{t_i} \in [0,K_1]} | \mathcal{F}_{t_0} \right] = \int_0^{+\infty} I_{x \in [0,K_1]}\, q_{X_{ti}|\mathcal{F}_{t0}}(x) dx$$

Now, define $\varphi^1(u)$ as the Fourier transform of the function $I_{x \in [0,K_1]}$, $x \in \mathbb{R}^+$:

$$\varphi^1(u) = \int_{-\infty}^{+\infty} I_{x \in [0,K_1]} e^{iux} dx.$$

It can be easily checked that this last integral is indeed equal to eq. (29). The function $I_{x \in [0,K_1]}$ can be retrieved by inverting the Fourier transform $\varphi^1(u)$:

$$\begin{aligned} I_{x \in [0,K_1]} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi^1(u) e^{-iux} du \\ &= \frac{1}{\pi} \int_0^{+\infty} \varphi^1(u) e^{-iux} du, \end{aligned} \qquad (31)$$

where the second equality results from the symmetry of the integrand, which is itself due to the fact that the function $I_{x \in [0,K_1]}$ is real (no imaginary component). The combination of eq.(30) and eq.(31) allows us to infer that:

$$\begin{aligned} \mathbb{E}\left[I_{X_{t_i} \in [0,K_1]} | \mathcal{F}_{t_0} \right] &= \frac{1}{\pi} \int_0^{+\infty} \int_0^{+\infty} \varphi^1(u) e^{-iux}\, q_{X_{ti}|\mathcal{F}_{t0}}(x) dx du \\ &= \frac{1}{\pi} \int_0^{+\infty} \varphi^1(u) \mathbb{E}\left(e^{-iuX_{t_i}} | \mathcal{F}_{t_0} \right) du \end{aligned} \qquad (32)$$

The integrand of this last equation contains the Laplace transform of the aggregated losses process. This can be worked out as follows

$$\mathbb{E}\left(e^{-iuX_{t_i}}|\mathcal{F}_{t_0}\right)=e^{-iuX_{t_0}}\mathbb{E}\left(e^{-iu\sum_{i=N_{t0}}^{N_{ti}}Y_i}|\mathcal{F}_{t_0}\right)$$

$$=e^{-iuX_{t_0}}\sum_{k=0}^{+\infty}P(N_{t_i}-N_{t_0}=k)\left(\mathbb{E}\left(e^{-iuY}\right)\right)^k. \tag{33}$$

Proposition 4.2 *The following expectation may be rewritten as follows:*

$$\mathbb{E}\left[\left(K_2-X_{t_i}\right)I_{X_{t_i}\in(K_1,K_2]}|\mathcal{F}_{t_0}\right]=$$

$$\frac{1}{\pi}\int_0^{+\infty}\varphi^2(u)e^{-iuX_{t_0}}\sum_{k=0}^{+\infty}P(N_{t_i}-N_{t_0}=k)\left(\mathbb{E}\left(e^{-iuY}\right)\right)^k du \tag{34}$$

where $\varphi^2(u)$ *is the Fourier transform of the function* $\left(K_2-x\right)I_{x\in[K_1,K_2]}$, $x\in\mathbb{R}^+$,

$$\varphi^2(u)=\frac{K_1}{iu}e^{iuK_1}-\frac{K_2}{iu}e^{iuK_1}+\frac{1}{(iu)^2}\left(e^{iuK_2}-e^{iuK_1}\right) \tag{35}$$

The probabilities $P(N_{t_i}-N_{t_0}=k)$ *can be retrieved from proposition 2.4 whereas the expectation* $\mathbb{E}\left(e^{-iuY}\right)$ *is the Laplace transform of the claim size* Y *, valued at* iu *.*

Proof. The proof is analogous to the proof of proposition 4.1. It may be checked quickly that $\varphi^2(u)$, the Fourier transform of the function $\left(K_2-x\right)I_{x\in[K_1,K_2]}$, $x\in\mathbb{R}^+$,

$$\varphi^2(u)=\int_{-\infty}^{+\infty}\left(K_2-x\right)I_{x\in[K_1,K_2]}e^{iux}dx,$$

is well equal to eq.(35).

The calculation of integrals (28) and (34) is done numerically by the Fast Fourier Transform algorithm. The FFT algorithm computes in only $O(n\log n)$ operations, for any input array $\{IN(j): j=0,\dots,NS-1\}$, the following output array:

$$OUT(m)=\sum_{j=0}^{NS-1}e^{-\frac{2\pi i}{NS}mj}.IN(j) \qquad m=0,\dots,NS-1$$

The first step to use this numerical method, consists in discretizing the integrals (28) and (34). We note Δu , the step of discretization and NS, the number of steps. The mesh of discretization is defined as follows:

$$\{u_j\}=\{(j+1)\Delta u\in\mathbb{R}^+|\,j=0,...,NS-1\}$$

Next, we define a discretization mesh for the values of X_{t_0} , spaced by Δx , and counting the same number NS of elements as $\{u_j\}$ (this is a necessary condition to use the FFT algorithm)).

$$\{x_m\} = \{(m+1)\Delta x \in \mathbb{R}^+ | m = 0,...,NS-1\}$$

On the condition that steps of discretization Δu and Δx satisfy the equality:

$$\Delta u \Delta x = \frac{2\pi}{NS},$$

the discrete versions of equalities (28) and (34), for all $x_{m=0...NS-1}$, can be reformulated into suitable forms for the FFT algorithm as follows:

$$\frac{\pi}{BN} \underbrace{\mathbb{E}\left[I_{X_{t_i} \in [0,K_1]} | \mathcal{F}_{t_0}, X_{t_0} = m\Delta x \right]}_{OUT_1(m)} \approx$$

$$\sum_{j=0}^{NS-1} e^{-i\frac{2\pi}{NS} jm} \underbrace{\varphi^1(u_j) \sum_{k=0}^{NJ} P(N_{t_i} - N_{t_0} = k) \left(\mathbb{E}\left(e^{-iu_j Y} \right) \right)^k \Delta u}_{IN_1(j)} \tag{36}$$

$$\pi \underbrace{\mathbb{E}\left[\left(K_2 - X_{t_i} \right) I_{X_{t_i} \in (K_1, K_2]} | \mathcal{F}_{t_0}, X_{t_0} = m\Delta x \right]}_{OUT_2(m)} \approx$$

$$\sum_{j=0}^{NS-1} e^{-i\frac{2\pi}{NS} jm} \underbrace{\varphi^2(u_j) \sum_{k=0}^{NJ} P(N_{t_i} - N_{t_0} = k) \left(\mathbb{E}\left(e^{-iuY} \right) \right)^k \Delta u}_{IN_2(j)} \tag{37}$$

where NJ is an upper bound chosen such that the probability of observing NJ claims in the interval of time $[t_0, t_i]$ is negligible. The left hand terms of equations (36) and (37) are the output vectors computed by a standard FFT algorithm. By combining them, one can calculate the expected values of future nominal, for a wide range of initial values for X_{t_0}:

$$\mathbb{E}\left(BN_{t_i} | \mathcal{F}_{t_0}, X_{t_0} = m\Delta x \right) = \frac{BN}{\pi} OUT_1(m) + \frac{1}{\pi} OUT_2(m)$$
$$\forall m = 1,...,NS-1 \tag{38}$$

Note that the calculation of the spread by the formula 23, at the issuance of the bond, only requires to determine the expected future nominal when $X_{t_0} = 0$. However, the knowledge of expected future nominal when $X_{t_0} > 0$, may be useful to reappraise an insurance bond, issued before t_0, and when some claims have already occurred.

5. NUMERICAL APPLICATIONS.

In this section, we apply the methodology introduced in previous sections to price insurance bonds on claims related to tornadoes hitting the US. As mentioned previously, a common practice adopted by actuaries is to assume that the parameters ruling the claims process are identical under P and Q. We have done the same and fitted the claims arrival process to monthly data running from 1974 to 2008. Those data have been presented in section 2, figure 0. The calibration is done by log-likelihood maximization. We have set the reversion parameter b to zero and upper bounded the speed of mean reversion a to 5. This bound is chosen to ensure that the intensity process λ_t does not become negative. The table 0 contains the parameters for the claims arrival process. The speed of mean reversion α reaches the bound of 5. The seasonality and the stochasticity embedded into the intensity of claims provides a better fit of data than the one obtained with a simple Poisson process. Indeed, the loglikelihood obtained by fitting a constant intensity Poisson process is worth -9124 which is below -6154, the log-likelihood obtained with our model.

Table 1: Parameters of the claims arrival process.

a	2.0000	δ	491.6078
b	0.0000	β	324.4812
σ	46.1072	γ	0.5954

From the proposition 2.4, we can infer the probability density functions of N_t after 3, 6, 9 and 12 months. Those densities are plotted in figure 2.

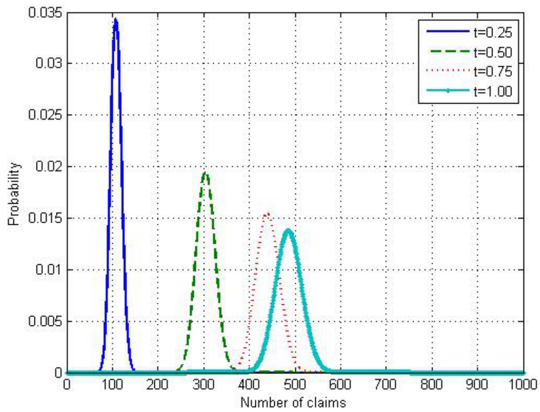

Figure 3: Distribution of claims by maturity.

From those densities, we can calculate the averages and standard deviations of the number of claims, after 3, 6, 9 and 12 months, which are presented in table 1. Per year, we foresee on average 486 tornadoes, and the 1 year volatility is around 29 tornadoes. From those figures, we also see that, most of tornadoes (on average 304-108=202) will hit the US during the second term.

Table 2: Means and deviations of N_t.

	$t = 3M$	$t = 6M$	$t = 9M$	$t = 12M$
$E(N_t)$	108.1816	304.5737	440.0100	486.2359
$\sigma(N_t)$	11.6358	20.4868	25.8611	28.9267

In this example, we have decided to compute the fair spreads that insurance bonds of maturities ranging from 3 months to 12 months should pay above the risk free rate, here set to 3%. The nominal, NB , is 80 millions. This nominal is decreased if the aggregated losses breach the trigger of 20 millions. The coupons are paid quarterly. Other bonds characteristics are summarized in table 2.

Table 3: Parameters of bonds.

r	3%	t_n	3M, 6M, 9M, 12M
K_1	20 Millions $	t_1	3M
K_2	100 Millions $	Δt	3M

The size of claims caused by each tornado that occurred between 1974 and 2008 has been retrieved on Sheldus. The exposure of the insurer issuing the insurance bonds has been assumed to be $1/20$ of the total claims cost. The size of claims is modeled by a Gamma random variable whose parameters are $\theta = 4.7511$ and $k = 0.01380$ (fitted by the method of moments). The average size of claims is of 0.065 million$ and the standard deviation is of 0.558 million $. The Gamma distribution is the most common distribution to model claims, but our approach is still efficient with another distribution, on the condition that its Laplace transform has a closed form expression. The parameters used for the FFT algorithm are provided in table 3. It seems that this choice leads to a good accuracy of calculations.

Table 4: FFT parameters.

NS	2^{14}	dx	$1million$	du	$\frac{2\pi}{N\,dx}$

The figure 3 presents the spreads ranked by bonds maturities (3, 6, 9 or 12 months), and for a set of initial values of aggregated losses, $X_{t=0}$. Without surprise, the higher is the initial value of the total claims, the higher is the spread. This is a direct consequence of the fact that the aggregated losses at time $t=0$ directly reduces the nominal. The spread is positively correlated with the maturity of the bond: the spreads of 3, 6, 9 and 12 months bonds respectively quote 0.7572% 2.6500% 4.5460% 4.3916%, when $X_0 = 0$.

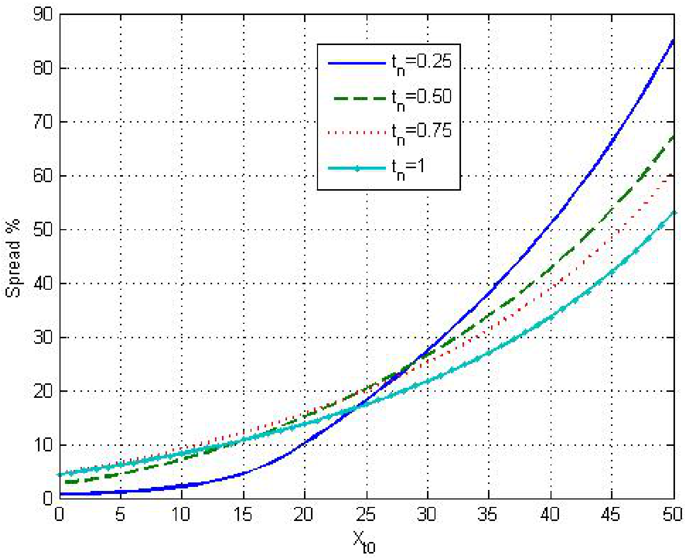

Figure 4: Spreads by maturities and by X_{t_0}.

Figure 4 presents the expected remaining nominal after 3, 6, 9 and 12 months. This expected nominal decreases with time given that the expected number of occurred claims rises with the time horizon. The expected nominal is also inversely proportional to the total initial amount of claims X_0. When X_0 tends to 100 millions \$, the nominal falls to zero (and the spread tends to infinity).

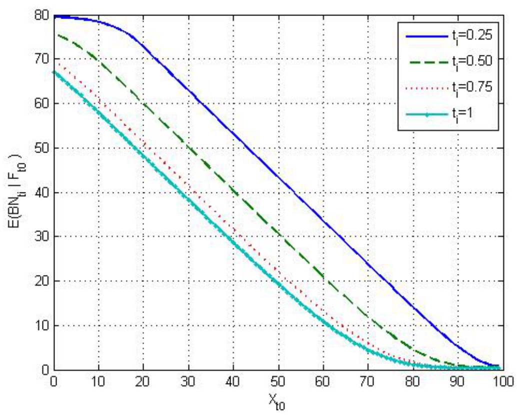

Figure 5: Expected remaining nominal by maturities and by X_{t_0} .

6. CONCLUSIONS.

This paper proposes a method to price catastrophe bonds paying multiple coupons, when the number of claims is under the influence of a stochastic seasonal effect. Modeling the seasonality is particularly important for insurance securities, whose valuation is related to claims with an intensity which is rising during certain periods of the year, as storms, hurricanes. Another important feature of this work is the presence of a mean reverting process, embedded in the intensity of the claims arrival process. This stochastic process may be calibrated so as to reflect the influence of climate changes on the frequency of claims. Despite the apparent complexity of the claims arrival process, we established a simple recursion to compute the probability distribution of the number of claims.

The insurance bond periodically pays a coupon equal to a constant percentage of the nominal, reduced by the amount of aggregated losses, exceeding a certain trigger level. At maturity, what is left of the nominal is repaid. In order to compensate for this eventual loss of nominal, the coupon rate always exceeds the risk free rate. The calculation of the spread above the risk free rate requires the appraisal of future expected remaining nominals. As no closed form expression exists for the expected future nominals, we showed how to compute them by the Fast Fourier Transform Algorithm. This approach is also shown to be an efficient method for the reappraisal of insurance bonds when some claims have occurred.

Catastrophe bonds offer an interesting alternative for investors who wish to diversify their exposure to risks, and this paper provides an efficient computational method to price those assets. Yet, many issues and uncertainties about the underlying assumptions remain unsolved. In particular, the shortcoming that consists to assume the independence between frequency and size of claims probably should be dropped. This point should be investigated in future research.

APPENDIX A.

This appendix details the dynamics of the model presented in this paper under the real measure, and the links between parameters defining the aggregated losses process under real and risk neutral measures. Let us note P the physical measure. Under P, the intensity of the claims arrival process is given by :

$$\lambda_t^P = \lambda^P(t) + \lambda_t^{OU,P} .$$

where $\lambda^P(t)$ is defined by three real constant parameters $\delta^P, \beta^P, \gamma^P \in \mathbb{R}$:

$$\lambda(t) = \delta^P + \beta^P \cos\left((t+\gamma^P)2\pi\right).$$

and where $\lambda_t^{OU,P}$, is an Ornstein Uhlenbeck process, whose the speed and level of mean reversion and the volatility are real constants, respectively denoted $a^P, b^P, \sigma^P \in \mathbb{R}$. The dynamic of $\lambda_t^{OU,P}$ is ruled by the following stochastic differential equation:

$$d\lambda_t^{OU} = a^P(b^P - \lambda_t^{OU})dt + \sigma^P dW_t^P \quad , \tag{39}$$

Under P, the size of claims are positive random variables of density $f_Y^P(y)$. The changes of measure from P to a risk neutral measure Q that leads to the model developed in section 2 and 3, are defined by the following Radon-Nikodym derivative (for details see e.g. Shreve 2004, chapter 11):

$$\frac{dQ}{dP} = \exp\left(\sum_{k=1}^{N_t} \ln(\kappa g(Y_k)) + \int_0^t \lambda_u (1-\kappa) du\right)$$

$$\exp\left(-\frac{1}{2}\int_0^t \xi^2 du - \int_0^t \xi dW_u^P\right) \tag{40}$$

where κ is a non-negative constant and ξ is here a real constant. The function $g(.)$ is measurable $g : \mathbb{R}^+ \to \mathbb{R}$ and satisfies the following relation:

$$\int_0^{+\infty} g(y) f^P(y) dy = 1$$

Equation (40) is the product of two exponentials. The first one is a Radon-Nikodym derivative that modifies both the frequency of the claims arrival process and the claims distribution. In particular, it multiplies the intensity by a constant κ ($\lambda_t^Q = \kappa\lambda_t^P$). Parameters of the deterministic trend modelling the seasonality are then

$$\delta = \kappa\delta^{P},$$
$$\beta = \kappa\beta^{P},$$
$$\gamma = \gamma^{P}.$$

The density function of claims under Q is distorted by the function $g(y)$ as follows:

$$f(y) = g(y) f^{P}(y)$$

The second exponential in (40) is a Radon-Nikodym derivative affecting only the Brownian dynamics of the intensity. More precisely, under Q, $dW_u = dW_u^P + \xi du$ is a Brownian motion. The random part of the intensity (taken into account the multiplicative factor κ) is solution of the next SDE:

$$d\lambda_t^{OU,Q} = a(b - \lambda_t^{OU,Q})dt + \sigma dW_t^Q \quad ,$$

where

$$a = a^{P}$$
$$b = \kappa\left(b^{P} - \frac{\sigma^{P}\xi}{a^{P}}\right)$$
$$\sigma = \kappa\sigma^{P}$$

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