MARGRABE OPTION AND LIFE INSURANCE WITH PARTICIPATION

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Abstract:
Life insurance rating is mainly based on two principles: discounting effect and mortality risk. The traditional way to price these contracts was to use a constant discount rate and a fixed life table. Nevertheless, evidences show that neither financial conditions nor mortality figures remain constant. If financial uncertainty is nowadays fully accepted and modelled quite a lot, longevity risk must be also considered and is of first importance for actuaries, especially in fair valuation of life insurance with profit.

The purpose of this paper is to present a model taking into account simultaneously financial and longevity risks and to apply it to the valuation of life insurance contracts with a participation scheme based on the global profit of the insurer. In particular, we show that a life insurance contract with global participation can be seen as a portfolio of zero coupons and Margrabe options.

Keywords: stochastic mortality, affine processes, participation scheme, fair valuation, Margrabe option

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1. INTRODUCTION

Fair valuation of life insurance liabilities is certainly for actuaries a major topic, especially in the context of the IAS/IFRS norms as well as in Solvency 2, where market consistent valuations of contracts must be considered. A lot of papers have been devoted these last years to the computation of fair valuation of life insurance contracts using an option approach (see Briys and De Varenne (1997)). Because of its importance in the insurance industry, a special focus has been made in the literature on contracts with participation scheme (also called life insurance with profit). These contracts are characterized by a technical interest rate guaranteed for the policyholder and a participation scheme given ex post depending on the profits of the insurer. Even if various options embedded in this kind of contract have been considered and valued (bonus option (e.g. Bacinello (2001), Hansen and Miltersen (2002), Devolder and Dominguez-Fabian (2005), Barbarin and Devolder (2005)), surrender options (e.g. Grosen and Jorgensen (2000), Bacinello (2001)), default risk (Bernard, Le Courtois and Quittard-Pinon (2005)), solvency risk (Grosen and Jorgensen (2002)), longevity risk is generally not taken into account especially in the bonus valuation. This could seem strange at first glance for actuaries! In our opinion, this fact could be explained mainly by two reasons. First, a lot of modern life insurance contracts in different markets are more financial products than classical life insurance; the influence of mortality becomes negligible. But even if indeed many new life products do not require sophisticated mortality models, the evolution of longevity remains an important issue for life actuaries, especially in pension and annuity business, and is still well present for a large part of existing portfolios.

A second reason was, till recently, a big contrast from a theoretical point of view between the variety of financial models for interest rates, equities, options,... and a lack of efficient stochastic time continuous models for mortality. But things have changed and different models of stochastic mortality have been proposed these last years (e.g. Dahl (2004), Biffis (2005), Luciano and Vigna (2005), Cairns, Blake and Dowd (2006), Schrager (2006)).

Mortality risk for an insurer has essentially two components : a diversifiable aspect linked with the law of large numbers and a non-diversifiable part linked with the general trend of longevity. Clearly the classical tools of life insurance cope with the first component and statistical arguments can be used. A natural hedging is then to have large portfolios of policy holders. From a financial point of view, the diversifiable aspect of the
mortality risk needs as such no risk premium and the physical probability measure can be used as well for pricing as for risk management perspective. This is the classical framework of life insurance. On the other hand the longevity risk induced by the collective improvement (or deterioration!) of mortality is clearly a residual risk that cannot be hedged by any statistical argument. In order to compute market prices of such liabilities driven by longevity risk, a risk premium must then be added to the best estimate. Several techniques can be used in order to compute such adjustment. We could use risk measure methods and consider for instance Wang transform techniques (see for instance Denuit, Devolder and Goderniaux (2007)) or just apply explicit risk margins (cf. Solvency 2). Another classical tool, perfectly coherent with finance, is the change of measure (see for instance Biffis, Denuit and Devolder (2010) or Luciano, Regis and Vigna (2012)). As pointed out by Biffis, Denuit and Devolder, the physical measure of mortality can be used for solvency analysis or realistic projections of cash flows; on the other hand, a change of probability measure can be considered in order to compute fair valuation or reserves based on longevity risk. Taking into account the combination of financial and longevity risks which interest us in this paper, we have chosen here to work with this change of measure technique. More precisely, we have developed mortality models that can be used as well in a physical probability as in a risk neutral perspective. This strong analogy between the longevity risk and classical financial risks has of course some practical limitations, especially in terms of calibration. Unlike the classical financial risk where we can extract a risk neutral measure from a huge set of real prices, there are not yet a lot of available market instruments based on mortality or longevity. As pointed out in several papers (see for instance Denuit, Devolder and Goderniaux (2007)), real prices of annuities as proposed by insurance can help us to estimate a level of risk premium and therefore a coherent change of probability (even if this is not a real financial market of annuities!). Despite these practical difficulties of calibration, we think that this framework is quite useful from a theoretical and prospective point of view and constitutes an objective way to integrate risk premiums in the computation of fair values. The future developments of mortality and longevity derivatives on the financial markets should then help us progressively to estimate adequate risk neutral corrections.

The purpose of this paper is to integrate some of these techniques of stochastic mortality in the valuation of life insurance contracts, especially in the bonus design and computation. Financial risk and longevity risk are then considered simultaneously in a
stochastic framework. In particular, different formulas of bonus are studied depending on the interactions between financial and longevity profits in the definition of the participation scheme. In this context, we show that a pure endowment contract with participation in the global profit of the insurer (taking into account investment performances together with mortality results) can be seen as a portfolio of zero coupons and Margrabe options (Margrabe (1978)). Explicit results of fair valuation are then given in a Gaussian environment.

The paper is organised as follows. Section 2 introduces the kind of contract to value and the different participation schemes based on financial profits and/or longevity profits. Section 3 introduces the financial market. In particular, the term structure of interest rates is modelled by the well-known one-factor Hull and White model. Section 4 presents the stochastic longevity model based on affine processes (Schrager (2006)). In Section 5, we obtain under the financial and longevity models explicit analytical formulae for the fair valuation of a pure endowment contract with participation. In particular, the technique of change of numeraire is successfully applied, introducing an new specific numeraire, called the endowment numeraire, and generalizing for life insurance the classical zero coupon bond prices. In Section 6 we relax the Gaussian hypothesis for modelling mortality rates. Finally Section 7 presents different numerical illustrations, while technical issues are developed in Appendix (Section 10).

2. LIFE INSURANCE CONTRACT WITH VARIOUS PARTICIPATION SCHEMES

We consider a pure endowment life insurance contract issued at time $t=0$, i.e. a contract paying a fixed amount of money in case of survival of the policyholder at some given maturity, and nothing in case of death before maturity. We will denote by:

1. $T$ the maturity of the contract;
2. $x$ the initial age (at $t=0$) of the policyholder;
3. $C$ the guaranteed capital paid in case of survival (paid at time $T$ if the insured is still alive, also called 'guaranteed benefit');
4. $\pi$ the lump sum paid at time $t=0$ by the policyholder. This lump sum is supposed to be invested in an investment fund.
Traditional actuarial techniques give the following expression for the premium $\pi$:

$$\pi = C \frac{T P_x}{(1+i)^T} = C T E_x$$

where $i$ is the technical rate and $T P_x$ denotes the survival probability of a person of initial age $x$ until age $x + T$.

In top of the guaranteed benefit $C$, a participation scheme is offered to the insured. The real benefit paid at time $T$, if the insured is still alive at age $x + T$, can be written as:

$$C(T) = C + P(T)$$

where $C$ is the guaranteed benefit and $P(T)$ is the terminal bonus.

The cash flow $C(T)$ is a random variable depending on the definition of the participation amount $P(T)$.

The basic principle of a participation scheme is to give back to the policyholder a part of the final surplus if any.

The profits at maturity are generated by two sources in this kind of contract:

1. financial profits: difference between the investment returns and the technical rate.
2. mortality profits: difference between the real mortality pattern and the initial survival probabilities used in the pricing.

From a theoretical point of view, we can introduce five different kinds of participation scheme depending on how these two sources of profit are interacting in the definition of the terminal bonus $P(T)$ paid to the policyholder:

Case 1: contract without any participation;
Case 2: contract with participation only in the financial profits;
Case 3: contract with participation only in the mortality profits;
Case 4: contract with participation in the financial and mortality profits without compensation;
Case 5: contract with participation in the global profit.

We are interested in the fair valuation of all these different kinds of contract. Even if some of these arrangements are quite unusual from a practical point of view,
a systematic valuation of all these cases will allow us to analyse the interaction between financial and longevity risks.

In order to model the two risks embedded in this product, we must introduce a financial market model and a stochastic mortality model.

In view of valuation, time continuous modelling will be used for both risks (assumed to be independent).

We consider a global filtered probability space \((\Omega, (\mathcal{H}_t)_{t\in[0,T]}, P)\) reflecting all the existing risks. We can then introduce two strict sub-filtrations of \((\mathcal{H}_t)_{t\in[0,T]}\):

1. a financial filtration : \(\mathcal{F}_t; t \in [0,T]\) modelling the accumulation of information coming from the financial markets;
2. a longevity filtration : \(\mathcal{G}_t; t \in [0,T]\) modelling the accumulation of information linked with the longevity evolution of the population.

The sample space \(\Omega\) appears as the product \(\Omega_1 \times \Omega_2\) of the financial sample space \(\Omega_1\) and the mortality sample space \(\Omega_2\).

3. THE FINANCIAL MARKET

The modelling of the financial market is based on two basic processes:

1. the term structure of interest rates;
2. the evolution of the investment fund.

Because we are interested in valuation tools, we will directly work in a risk neutral environment.

Under the risk neutral probability related to financial uncertainty and denoted by \(\mathcal{Q}\), we consider the following financial model:

- the term structure of interest rates is given by the one-factor Hull and White model. This well-known model has been chosen as example for the simplicity of presentation, but the same methodology can be easily applied for more general Gaussian models (e.g. the two-factors Hull and White model). The non Gaussian case is briefly investigated in Section 6.

In this model, the risk free rate \(r(t)\) is supposed to follow a stochastic process solution of the following equation:

\[
dr(t) = \left(\theta(t) - ar(t)\right)dt + \sigma_r dw_r(t)
\]  

(3)
where \( W_r \) is a standard Brownian motion adapted to the filtration under \( Q \), \( a \) and \( \sigma_r \) are positive constants and \( \theta(t) \) is a deterministic function.

In an arbitrage free framework, the zero coupon bond prices are then given by:

\[
P(t,s) = \mathbb{E}_Q \left[ \exp \left( -\int_t^s r(u) du \right) \right]
\]

(4)

where \( P(t,s) \) is the price at time \( t \) of a zero coupon with maturity \( s > t \).

Let us recall that in the Hull and White model, this price is explicitly given by:

\[
P(t,s) = \exp \left( A(t,s) - r(t) B(t,s) \right)
\]

(5)

with

\[
B(t,s) = \frac{1 - e^{-a(s-t)}}{a};
\]

\[
A(t,s) = \int_t^s \left( \theta(u), \frac{1 - e^{-a(s-u)}}{a}, \frac{1}{2} \sigma_r^2 \left( \frac{1 - e^{a(s-u)}}{a} \right)^2 \right) du.
\]

An explicit expression for the function \( A(t,s) \) and hence for the future price \( P(t,s) \) is generally obtained since when using Hull and White model for valuation purpose, the deterministic function \( \theta(t) \) is chosen so as to exactly fit the term structure of interest rates observed in the market at valuation date (cf. for instance Brigo and Mercurio (2006) for this analytical expression).

The zero-coupon bond price, seen as stochastic process, is also solution of the following stochastic differential equation:

\[
dP(t,s) = r(t) P(t,s) dt - \sigma(t,s) P(t,s) dw_r(t)
\]

(6)

with

\[
\sigma(t,s) = \frac{\sigma_r}{a} \left( 1 - e^{-a(s-t)} \right)
\]

- the evolution of the investment fund \( S(t) \) in which the initial lump sum is invested by the company is given by:

\[
dS(t) = S(t) \left[ r(t) dt + \sigma_S \left( \rho dw_r(t) + \sqrt{1 - \rho^2} dw_{S}(t) \right) \right]
\]

(7)
where $\sigma_s$ is a positive constant, $\rho$ is a constant (positive or negative), and $W_s$ is a standard Brownian motion adapted to the filtration $\mathcal{F}_t, t \in [0,T]$ under the risk-neutral measure $\mathbb{Q}$ and uncorrelated with the process $W_{\mu}$.

4. THE STOCHASTIC LONGEVITY MODEL

In order to model the uncertainty related to longevity, we introduce a stochastic intensity of mortality denoted by $\mu_x(t), t \in [0,T]$.

The process $\mu_x$ is now adapted to the "actuarial" filtration $\mathcal{G}_t$ and is supposed to be solution of the following affine equation:

$$d\mu_x(t) = (\xi(t) - b(t) \mu_x(t))dt + \sigma_{\mu} dW_{\mu}(t).$$

In this class of processes, we can consider as examples:

1. non mean reverting processes (Luciano / Vigna): $\xi = 0, b = -b > 0$

$$d\mu_x(t) = \beta \mu_x(t)dt + \sigma_{\mu} dW_{\mu}(t)$$

2. Hull and White mortality model: $\xi > 0, b > 0$

$$d\mu_x(t) = (\xi(t) - b\mu_x(t))dt + \sigma_{\mu} dW_{\mu}(t)$$

The process $W_{\mu}$ is a standard Brownian motion under the historical measure $\mathbb{P}$ and adapted to the filtration $\mathcal{G}_t$.

Once again, we could have used more general Gaussian models. We can introduce the survival function generalizing the classical survival probability (1) in the case of a stochastic mortality:

$$p_x(t,s) = \mathbb{E}_p \left[ \exp \left( -\int_t^s \mu_x(u)du \right) \right | \mathcal{G}_t]$$

Explicit formulations of this survival function at time $t$ can be easily obtained exactly like zero coupon bond prices in Section 3:

$$p_x(t,s) = \exp(A(t,s,x) - B(t,s,x)\mu_x(t))$$

with

$$B(t,s,x) = \int_t^s \exp^{t_u} du$$

and

$$A(t,s,x) = \int_t^s (\xi(u)B(u,s,x) - \frac{1}{2} \sigma_{\mu}^2 B^2(u,s,x))du.$$
This relation shows that in an affine model, the survival process between times \( t \) and \( s \), generalizing the classical survival probability between these instants, only depends on the stochastic intensity of mortality at time \( t \).

Remark that in the case of Hull and White model for mortality, the function \( B(t,s,x) \) has an analytical expression given by:

\[
B(t,s,x) = \frac{1}{b}[1-e^{-b(s-t)}],
\]

and the function \( A(t,s,x) \) has an analytical expression depending on the initial survival function \( p_x(0,) \).

This survival process is also solution of a stochastic differential equation:

\[
dp_x(t,s) = \mu_x(t)p_x(t,s)dt - \chi(t,s)p_x(t,s)dw_{\mu}(t)
\]

where:

\[
\chi(t,s) = \sigma_{\mu}B(t,s,x)
\]

As in the financial model, we can introduce a risk neutral version of this stochastic mortality framework, by considering a market price of longevity risk denoted by \( \lambda \).

The new probability measure \( \overline{P} \) is by definition the measure such that the stochastic process:

\[
\overline{w}_{\mu}(t) = w_{\mu}(t) + \int_{0}^{t}\lambda(s,x_{\mu}(s))\,ds
\]

is a standard Brownian motion. We refer the reader to the well know Girsanov theorem (see for instance Brigo and Mercurio (2006)).

Under \( \overline{P} \), equation (10) can be written:

\[
d\mu_{x}(t) = (\xi(t) - b(t)\mu_{x}(t))\,dt + \sigma_{\mu}\overline{w}_{\mu}(t)
\]

For particular forms of the market price of longevity risk, this model will have the same form as under the historical measure (10):

\[
d\mu_{x}(t) = (\overline{\xi}(t) - \overline{b}(t)\mu_{x}(t))\,dt + \sigma_{\mu}\overline{w}_{\mu}(t)
\]

where:

\[
\lambda(t,x_{\mu}(t)) = \lambda_1(t) + \lambda_2(t)\mu_{x}(t)
\]

with:

\[
\lambda_1(t) = \frac{\xi(t) - \overline{\xi}(t)}{\sigma_{\mu}}
\]

(correction of the moving target),
\[ \lambda_2(t) = \frac{\frac{\partial b(t)}{\partial t} - b(t)}{\sigma} \]

(correction of the speed of convergence).

We can introduce then a risk neutral survival function under the adjusted probability measure \( \bar{P} \) as follows:

\[ \bar{p}_x(t,s) = \mathbb{E}_{\bar{P}} \left[ \exp \left( - \int_t^s \mu_x(u) \, du \right) \right] \]  

(16)

Positive values of the market price of longevity risk lead in the risk neutral world, to lower values of the mortality intensity and bigger values of the survival function, which can be interpreted as a safety principle for pure endowment contracts.

This risk neutral survival function has exactly the same form as the original survival function:

\[ \bar{p}_x(t,s) = \exp \left( \bar{A}(t,s,x) - \bar{B}(t,s,x) \mu_x(t) \right) \]  

(17)

with

\[ \bar{B}(t,s,x) = \int_t^s \exp \left( \int_t^v (\frac{\partial b(v)}{\partial v} + \lambda_2(v) \sigma) \, dv \right) \, du \]

\[ \bar{A}(t,s,x) = -\int_t^s \left( \xi(u) \bar{B}(u,s,x) - \frac{1}{2} \sigma^2 \bar{B}^2 (u,s,x) \right) \, du \]

with \( \xi(u) = \xi(u) - \lambda_2(u) \sigma \).

And the stochastic differential equation now becomes:

\[ d\bar{p}_x(t,s) = \mu_x(t) \bar{p}_x(t,s) \, dt - \chi(t,s) \bar{p}_x(t,s) \, d\bar{w}_\mu(t) \]  

(18)

where \( \chi(t,s) = \sigma \bar{B}(t,s,x) \).
5. **VALUATION OF THE CONTRACT**

5.1 **General methodology**

We want to valuate the contract (2) at each moment \( t \) \((0 \leq t \leq T)\). The final cash flow \( C(T) \) is a \( \mathcal{H}_T \) measurable random variable.

Using standard methods of fair valuation, we can estimate the fair value of the contract by a conditional expectation:

\[
FV(t) = \mathbb{E}_{\tilde{Q}} \left[ C(T) \exp \left( - \int_t^T r(s) \, ds \right) \exp \left( - \int_t^T \mu_x(s) \, ds \right) \mid \mathcal{H}_t \right]
\]  

(19)

where

- \( \tilde{Q} = Q \times P^* \) is a martingale measure on the product space \( \Omega = \Omega_1 \times \Omega_2 \);
- \( Q \) is the financial risk neutral measure;
- \( P^* \) is a risk neutral measure equivalent to the historical longevity measure \( P \).

For this longevity measure \( P^* \), two cases can be considered:

1. a "pure" fair valuation principle: \( P^* = P \), the historical measure.
2. a "loaded" fair valuation principle: \( P^* = \tilde{P} \), the risk neutral measure (cf Section 4).

We will develop the pricing formula in this context.

From the asset side, the asset available at maturity for each contract still in force at time \( T \), taking into account the evolution of the investment fund and the number of survivors, is given by:

\[
A(T) = \frac{\pi \cdot S(T)}{\exp \left( - \int_0^T \mu_x(s) \, ds \right)}
\]

(20)

From the liability side, we will consider the five cases introduced in Section 2 above, corresponding to different types of participation.

Let us recall that in Case 1, the insurance company only pays to the policyholder the fixed capital amount \( C \), whatever the real return of the investment fund.

Case 2 corresponds to an additional participation in the financial profits on
investments of the company. This is actually the traditional way most companies attribute their profit sharing.

Case 3 is less usual and more academic, but it is presented here as it will be further used in Cases 4 and 5.

The fourth and fifth cases suppose that the policyholder participates in all types of profits of the company: the financial profits on its investments, and the mortality profits. Case 4 supposes no compensation between these two categories of profits, while Case 5 does.

5.2 Case 1: Contract without participation

For a contract without any participation, the liability cash flow \( C(T) \) is deterministic.

By using classical properties of the conditional expectation with respect to a \( \sigma \)-algebra, by definition of \( \mathcal{H}_t \) and by the independence assumption between financial and mortality risks, we obtain for the fair value of the contract:

\[
FV(t) = \mathbb{E}_Q \left[ C \exp \left( -\int_t^T r(s) ds \right) \exp \left( -\int_t^T \mu_x(s) ds \right) \mid \mathcal{H}_t \right] 
\]

\[
= C \mathbb{E}_Q \left[ \exp \left( -\int_t^T r(s) ds \right) \right] \mathfrak{I}_t \mathbb{E}_P^\mathfrak{I}_t \exp \left( -\int_t^T \mu_x(s) ds \right) G_t 
\]

\[
= CP(t,T) \overline{F}_x(t,T) 
\]

\[
= FV^B(t) 
\]

(fair value of the basic contract).

Taking into account the affine structure of the zero coupon bond price and of the survival probabilities, we get explicitly:

\[
FV(t) = FV^B(t) = \exp \left[ A(t,T) + \overline{A}(t,T,x) - r(t)B(t,T) - \mu_x(t) \overline{B}(t,T,x) \right] 
\]
5.3 Case 2: Contract with financial participation

In a contract with participation only in the financial profits, the insurer gives back to the policy holder a part of the difference between the real return on asset at maturity and the technical guaranteed rate \( i \).

The total cash flow at time \( T \) is then given by

\[
C(T) = C + P^f(T)
\]

with

\[
P^f(T) = \eta C \left[ \frac{S(T)}{(1+i)^T} - 1 \right]
\]

and \( \eta \) is the participation rate \( (0 < \eta \leq 1) \).

The fair value becomes:

\[
FV(t) = \mathbb{E}_Q \left[ (C + P^f(T)) \exp \int_t^T r(s)ds \exp \int_t^T \mu_s(s)ds \right]
\]

\[
= FV^B(t) + \eta \frac{C}{(1+i)^T} \mathbb{E}_Q \left[ (S(T) - (1+i)^T)^+ \exp \int_t^T r(s)ds \right]
\]

\[
= FV^B(t) + \eta \frac{C}{(1+i)^T} \mathbb{E}_p \left[ \frac{d_{g_1}(t) + \eta \frac{C}{(1+i)^T} \mathbb{E}_Q \left[ (S(T) - (1+i)^T)^+ \exp \int_t^T r(s)ds \right]}{(1+i)^T} \right]
\]

The last expectation in this expression is the price of a call option on the investment fund with a strike corresponding to the guarantee.

Using the general pricing of a call under stochastic interest rates, the fair value can be written as:

\[
FV(t) = FV^B(t) + \eta \frac{C}{(1+i)^T} \mathbb{E}_p \left[ S(t) \Phi(g_1(t)) - (1+i)^T \Phi(g_2(t)) \right]
\]

where
\[ g_1(t) = \frac{\ln S(t) / (1 + i)^T - \ln P(t, T) + \frac{1}{2} m^2(t, T)}{m(t, T)} \]

\[ g_2(t) = \frac{\ln S(t) / (1 + i)^T - \ln P(t, T) - \frac{1}{2} m^2(t, T)}{m(t, T)} \]

with

\[ m^2(t, T) = \sigma_S^2(T - t) + \frac{1}{a^2} \int_t^T \sigma_r^2(u, T) du + \frac{1}{2} \rho \sigma_S \sigma(u, T) du. \]

In the particular case of Hull and White model for interest rates, this volatility function \( m(t, T) \) can be calculated explicitly as:

\[ m^2(t, T) = \sigma_S^2(T - t) + \frac{1}{a^2} \left( \frac{3}{2} + \frac{1}{2} e^{-2a(T-t)} - 2e^{-a(T-t)} \right) \]

\[ - \frac{2\rho \sigma_S \sigma_r}{a} (T-t) + \frac{2\rho \sigma_S \sigma_r}{a^2} (1 - e^{-a(T-t)}). \]

5.4 Case 3: Contract with mortality participation

The purpose of the participation is then to correct the difference between the life table and the real survival probabilities:

\[ C(T) = C + P^m(T) \]

with

\[ P^m(T) = \eta C \left[ \frac{\sum_{x=0}^T P_x \exp \left( -\int_0^T \mu_x(s) ds \right)}{\exp \left( -\int_0^T \mu_x(s) ds \right)} - 1 \right] \]

and \( \eta \) is the participation rate \((0 < \eta \leq 1)\)

The fair value becomes then:
$$FV(t) = \mathbb{E}_Q \left[ (C + P^m(T)) \exp \left( -\int_t^T r(s) ds - \int_t^T \mu_x(s) ds \right) | \mathcal{H}_t \right]$$

$$= FV^B(t) + \eta C P(t,T) \mathbb{E}_Q \left[ \left( \frac{\int_0^T \mu_x(s) ds}{\mathcal{H}_t} \right)^+ \exp \left( -\int_0^T \mu_x(s) ds \right) \right]$$

$$= FV^B(t) + \eta C \frac{P(t,T)}{P_X} \mathbb{E}_Q \left[ \left( \exp \left( -\int_0^T \mu_x(s) ds \right) - \frac{1}{T} \right)^+ \exp \left( -\int_t^T \mu_x(s) ds \right) \right]$$

Exactly like in Section 5.3, this last expectation can be interpreted as a call option.

In this case, the underlying asset can be seen as the saving account under the mortality rates:

$$B(t) = \exp \left( \int_0^T \mu_x(s) ds \right).$$

(25)

Taking into account the general form of the price of a call under stochastic interest rates, the fair value becomes:

$$FV(t) = FV^B(t) + \eta C \frac{P(t,T)}{T P_X} \left[ B(t) \Phi \left( l_1(t) \right) - \frac{P_X(t,T)}{T P_X} \Phi \left( l_2(t) \right) \right]$$

(26)

where:

$$l_1 = \frac{-\ln B(t) T p_X - \ln \frac{p_X(t,T)}{T p_X} + \frac{1}{2} \nu^2(t,T)}{\nu(t,T)}$$

$$l_2 = \frac{-\ln B(t) T p_X - \ln \frac{p_X(t,T)}{T p_X} - \frac{1}{2} \nu^2(t,T)}{\nu(t,T)}$$

with $$\nu^2(t,T) = \int_t^T (u,T) du.$$
In the particular case of Hull and White model for mortality rates, this volatility function \( n(t,T) \) can be calculated explicitly as:

\[
\begin{align*}
n^2(t,T) &= \frac{\sigma^2}{b^2} (T-t) + \frac{\sigma^2}{b^3} \left( -\frac{3}{2} - \frac{1}{2}e^{-2b(T-t)} + 2e^{-b(T-t)} \right).
\end{align*}
\]

5.5 Case 4: contract with participation in the financial and mortality profits without compensation

In order to mix the financial and the mortality results, a first approach is to give to the client the sum of the financial participation as computed in Section 5.3 and of the mortality participation as computed in Section 5.4. This means in practice that the contract is very generous for the policy holder; there is no compensation between the two risks.

The total cash flow at time \( T \) is then:

\[
C(T) = C + P^f(T) + P^m(T)
\]

where

- \( P^f(T) \) is the financial participation given by (22),
- \( P^m(T) \) is the mortality participation given by (24).

5.6 Case 5: Contract with participation in the global profits

Another way to mix financial and mortality profits is to consider the global result of the product. This means now that we accept compensation between the two sources of profits. For instance, profits in the financial part can be compensated by mortality losses.

The total cash flow to be paid at time \( T \) becomes then:

\[
C(T) = C + P(T)
\]

where \( P(T) \) is the total participation in the global results of the insurer and is given by:

\[
P(T) = \eta C \left[ \frac{\int_0^T P_x S(T) - 1}{\exp\left( -\int_0^T \mu_x(s) ds \right) (1+i)^T} \right].
\] (27)

The fair value is then given by:
\[ FV(t) = \mathbb{E}_Q \left[ \left( C + P(T) \right) e^{-\int_t^T (r(s)+\mu_x(s)) \, ds} \bigg| \mathcal{H}_t \right] \]

\[ = FV^B(t) + FV^P(t) \]

where \( FV^B(t) \) is given by (21), and \( FV^P(t) \) is the fair value of the participation liability.

This last fair value can be written:

\[ FV^P(t) = \eta \mathbb{E}_Q \left[ \left( \frac{T P_x S(T)}{\exp \int_0^T \mu_x(s) \, ds} - 1 \right)^+ \right. \]

\[ \left. \frac{-\int_T^T (r(s)+\mu_x(s)) \, ds}{(1+i)^T} \bigg| \mathcal{H}_t \right] \]

This last expectation can be interpreted as a Margrabe option under stochastic interest rates:

\[ \mathbb{E}_Q \left[ \left( \frac{T P_x S(T)}{\exp \int_0^T \mu_x(s) \, ds} - 1 \right)^+ \right. \]

\[ \left. \frac{-\int_T^T (r(s)+\mu_x(s)) \, ds}{(1+i)^T} \bigg| \mathcal{H}_t \right] \]

\[ = \frac{T P_x}{(1+i)^T} \mathbb{E}_Q \left[ \left( \frac{S(T)}{\exp \int_0^T \mu_x(s) \, ds} - \frac{(1+i)^T}{T P_x} \right)^+ \right. \]

\[ \left. \frac{-\int_T^T r(s) \, ds}{\exp \int_0^T \mu_x(s) \, ds} \bigg| \mathcal{H}_t \right] \]
The last expectation is the price of a Margrabe exchange option between the investment fund and a mortality index. In order to compute this expectation, we will introduce a new numeraire, generalizing the classical actuarial price of a pure endowment.

In a deterministic model, this price at time $t$ is given by:

$$P_{X+t} = \frac{1}{(1+i)^{T-t}} P_{X+t}$$

In a stochastic environment, we define a stochastic process: the pure endowment numeraire, given by

$$E_{X}(t, T) = E_{Q}\left[\exp^{-\int_{t}^{T} (r(s) + \mu_x(s))ds} | \mathcal{H}_t}\right]$$

This process can also be interpreted as a "generalized" zero coupon bond price where the risk free rate is adjusted by the mortality credit:

$$E_{X}(t, T) = E_{Q}\left[\exp^{-\int_{t}^{T} \bar{r}(s)ds} | \mathcal{H}_t}\right]$$

where: $\bar{r}(s) = r(s) + \mu_x(s)$.

We introduce also a new process $R(t)$, which will play the role of underlying asset:
\[ R(t) = S(t) \exp \left( \int_{0}^{t} \mu_{x}(s) ds \right) = S(t) B(t). \] (30)

The fair value of the participation becomes then:

\[
FV^P(t) = \eta C \mathbb{E}_{\tilde{Q}} \left[ \left. R(T) T E_x - 1 \right| \tilde{H}_t \right] \exp \left( - \int_{t}^{T} \tilde{r}(s) ds \right) \left| \tilde{H}_t \right]
\]

\[
= \eta C T E_x \mathbb{E}_{\tilde{Q}} \left[ \left. \left( R(T) - \frac{1}{T E_x} \right)^+ \right| \tilde{H}_t \right] \exp \left( - \int_{t}^{T} \tilde{r}(s) ds \right) \left| \tilde{H}_t \right]
\]

This expectation is the price of a call option on the new underlying asset \( R \), using the adjusted risk free rates \( \tilde{r} \).

Under the risk neutral measure, the new numeraire and underlying asset can be written as:

- **numeraire:**
  \[ dE_x(t, T) = E_x(t, T) \tilde{r}(t) dt - E_x(t, T) \Psi(t, T) dw_E(t) \]

where:

\[
\Psi(t, T) = \left( \sigma^2(t, T) + \chi^2(t, T) \right)^{1/2}
\]

\( w_E \) is a standard Brownian motion given by

\[
dw_E(t) = \frac{\sigma(t, T)}{\Psi(t, T)} dw_r(t) + \frac{\chi(t, T)}{\Psi(t, T)} dw_\omega(t)
\]

- **underlying asset:**

\[
dR(t) = R(t) \tilde{r}(t) dt + R(t) \sigma_x dw_x(t)
\]

where:

\[
dw_x(t) = \rho dw_r(t) + \sqrt{1 - \rho^2} dw_x(t)
\]

and \( \langle dw_E(t), dw_\omega(t) \rangle = \rho \frac{\sigma(t, T)}{\Psi(t, T)} dt \)
Using the general pricing of a call under stochastic interest rates, we can obtain the fair value of the participation:

\[
FV^P(t) = \eta C_T E_x \left[ R(t) \Phi(h_1(t)) - \frac{1}{T} E_x(t,T) \Phi(h_2(t)) \right] 
\] (31)

where

\[
h_1(t) = \frac{\ln R(t) T E_x - \ln E_x(t,T) + \frac{1}{2} V^2(t,T)}{V(t,T)}
\]

\[
h_2(t) = \frac{\ln R(t) T E_x - \ln E_x(t,T) - \frac{1}{2} V^2(t,T)}{V(t,T)}
\]

with:

\[
V^2(t,T) = \int_T^T \sigma^2_s (u,T) + 2 \Psi(u,T) \sigma_s \rho \sigma(u,T) \Phi(u,T) du
\]

\[
= \sigma^2_s (T-t) + \int_t^T \sigma^2_s (u,T) du + 2 \int_t^T \rho \sigma_s \sigma(u,T) du + \int_t^T \sigma_s (u,T) du
\]

\[
= m^2 (t,T) + n^2 (t,T)
\]

6. RELAXING THE GAUSSIAN HYPOTHESIS FOR MORTALITY

The normality hypothesis for mortality rates is used for getting (23). Now, it may seem reasonable to require to mortality models to lead only to positive mortality rates. The same remark also holds for interest rates, but to a smaller extent. In this section, we give indications on how relaxing this hypothesis.

We suppose that mortality rates follow a square root volatility model, namely the so-called CIR++ model (see Brigo and Mercurio (2006)):

\[
\mu_s(t) = x(t) + \phi(t,k,\theta,\sigma,\sigma_0)
\]

where the process \( x(t) \) is supposed to follow a CIR process:

\[
dx(t) = k(\theta - x(t)) dt + \sigma_s \sqrt{x(t)} dw(t), \quad x(0) = x_0
\]

where \( \phi \) is a deterministic function depending on the initial survival function, and \( k, \theta, \sigma \).
are positive constants with $2k\theta>\sigma_\mu^2$.\(^1\)

It is well known that the process $(x_t)$ has a noncentral chi-square distribution, and that the survival function has an affine term-structure, i.e. an expression similar to equation (12) above.

This model is generally called CIR++ model, and appears as a generalised CIR model, like Hull-White is a generalisation of Vasicek model for interest rates.

This model is commonly used for modelling interest rates for valuation purpose, especially in periods of volatile but very low rates in the market, as well as for modelling credit rates. This model is further developed for instance in Brigo and Mercurio (2006).

A risk neutral version of this model is once again easily introduced by considering the same expression for function $\lambda(t,\mu_x(t))$:

$$\lambda(t, x(t)) = \lambda_1(t) + \lambda_2(t)\sqrt{x(t)}$$

with

$$\lambda_1(t) = \frac{k}{\sigma_\mu}\sqrt{x(t)},$$

$$\lambda_2(t) = \frac{k - k_t}{\sigma_\mu}.$$  

In this case, the process $x(t)$ has the same analytical expression under the risk neutral measure with parameters $\bar{k}, \bar{\theta}$.

The stochastic differential equation (18) for the survival function under the risk neutral measure now becomes:

$$dp_x(t, T) = \mu_x(t)p_x(t, T)dt - B(t, T)p_x(t, T)\sigma_\mu \sqrt{r(t)} - \phi(t)dw_\mu(t)$$

(32)

where $B(t, T)$ is equal to some explicit function:

$$B(t, T) = \frac{2(e^{h(T-t)} - 1)}{2h + (k + h)(e^{h(T-t)} - 1)}$$

where $h = \sqrt{k^2 + 2\sigma_\mu^2}$. The stochastic differential equation for $p_x(t, T)$ can be rewritten completely in function of $p_x(t, T)$ (cf. Brigo and Mercurio (2006)).

\(^1\) This condition ensures that the origin is inaccessible to the reference CIR model so that the process $x(t)$ remains positive.
First, cases 1 and 2 are very similar to the Gaussian case. The reasoning of case 3 can be applied here as well, and analytical formulae for the expression:

\[
\mathbb{E}_t \left[ \exp \left( \int_0^T \mu_x(s) ds \right) - \frac{1}{P_x} \right]^{+} \exp \left( -\int_0^T \mu_x(s) ds \right) | G_t \]
\]

can still be obtained by using results for the pricing of zero-coupon calls and puts in the CIR framework (cf. Brigo and Mercurio (2006) Chapters 3 and 4, and Cox, Ingersoll and Ross (1985)). The expression contains now the cumulative distribution function of a noncentered chi-square distribution instead of the standard normal cumulative distribution. Remark that in the case interest rates also follow a CIR++ model, cases 2 and 3 can be extended, and analytical formulae are replaced by the ones prevailing for these models (cf. Brigo and Mercurio (2006) and Cox, Ingersoll and Ross (1985)).

Case 4 is directly derived, as in the Gaussian case, from cases 2 and 3.

Case 5 is less straightforward. Indeed, the complete reasoning can be performed up to (31) excluded, but a simple analytical expression for the Margrabe exchange option like in (31) cannot be obtained anymore. Actually, we can obtain an analytical expression, up to the calculation of a double integral.

If moreover interest rates also follow a CIR++ process, this double integral can be easily numerically calculated by using an adequate change of variable that leads to a one-dimensional integral, (cf. Chen and Scott (1992)). This means however that we do not get an analytical formula anymore, even if interest rates follow a square root volatility model.

Now, an explicit analytical approximation can be obtained by an analytical expression like in the Gaussian case by using the Gaussian mapping proposed in Brigo and Alfonsi (2005). The idea of this approximation is to approximate the CIR++ model for mortality by a Vasicek or Hull and White model with identical mean and variance.

7. NUMERICAL EXAMPLE

In this section, we calculate explicitly in the Gaussian case the fair value of a life insurance contract with different participations schemes by using the explicit formulae obtained in the previous sections.

We consider a pure endowment contract issued at time \( t = 0 \) with the following characteristics:
- initial age : \( x = 35 \) years;
- maturity of the contract : \( T = 30 \) years;
- benefit insured : \( C = 1,000,000 \) monetary units (paid at time \( T \) if the insured is still alive);
- first order actuarial bases :
  - technical rate : \( i = 2.50\% \);
  - technical life table : \( \{l_x\} \) determined by the Makeham constants\(^1\):
    \[ k=1000266.63, \quad s=0.999441704, \quad g=0.9999733441 \quad \text{and} \quad c=1.101077536. \]

One can see that the premium \( \pi \) of this contract under these hypotheses is equal to \( \pi = 432,218 \) monetary units.

The term structure of interest rates is modelled by a 1-factor Hull and White model with the following parameters: \( a = 2\% \), \( \sigma_r = 0.70\% \), which correspond to reasonable values that can be obtained from calibration on the interest rates options market. The yield curve at time \( t = 0 \) is the one deduced from the Euribor deposit and swap market rates at some recent date. In particular, the 30 years zero-coupon rate is equal to 3.41%.

The investment fund is supposed to be characterised by a volatility \( \sigma_S = 15\% \), while the instantaneous correlation between interest rates and the investment fund is supposed to be equal to \( \rho = 20\% \).

The mortality is supposed to be modelled by a 1-factor Hull and White model, with parameters \( b = 1\% \) and \( \sigma_\mu = 0.50\% \).

The experience mortality is supposed to be characterised by a survival function slightly different from the technical table (and corresponds to a survival function at \( t=0 \) of \( p_x(0,30) = 0.872630462 \) while the technical table leads to \( 35p_{30} = 0.906605814 \)).

The participation rate is supposed to be equal to \( \eta = 80\% \).

The fair valuation of the insurance contract is calculated at \( t = 0 \) under the previous financial and mortality modelling hypotheses, and by considering the different types of participation schemes. Results are the following:

\(^1\) This corresponds to the Belgian official MR table, for 1 million of births.
Fair Valuation FV(0) FV(participation)
Case 1 313,992 0
Case 2 456,314 142,322
Case 3 372,600 58,608
Case 4 514,922 200,930
Case 5 494,501 180,509

This table shows that under the current modelling hypotheses, the financial participation worth more than the mortality participation. The contract fair value without any participation is smaller than the contract premium (equal to 432,218). We also see that participation with compensation has less value than without compensation, which is not surprising. Now we see that the premium is less than the contract fair value in cases 2, 4 and 5, which means that a participation rate of 80% has been set too high by the company under the considered financial and underwriting modelling hypotheses.

The following results are obtained by replacing the guaranteed rate of 2.5% by a rate of 3%, hence closer to the 30-years rate of the market, which equals 3.4%.

Fair Valuation FV(0) FV(participation)
Case 1 313,992 0
Case 2 414,156 100,164
Case 3 372,600 58,608
Case 4 472,764 158,772
Case 5 457,657 143,665

The impact of this change is to make the option representing the financial participation closer to the money. Remark that the fair value without any participation does not change, as the cash-flow itself delivered at $T = 30$ does not change. The premium of the contract will however pass from 432,218 to 373,510 monetary units, which is not surprising as the guaranteed rate offered is higher. Once again, a participation of 80% appears too high in cases 2, 4 and 5 in view of the current modelling hypotheses.

The following tables illustrate the impact on the fair valuation of changes in the volatility of the investment fund (increase of the volatility in the market or of selected financial assets), as well as the impact of a negative correlation between interest rates and the investment funds¹.

¹ Remark that both positive and negative levels can be observed from historical data in the Euro zone, depending on the considered period, and depending on the type of investment fund.
From this table we see that the fair value of contracts with financial participation can increase potentially a lot in period of increasing volatility in the market. The sign of the correlation coefficient between interest rates and the investment assets also plays a role: less correlation decreases the fair value\(^1\).

The following tables show now the impact of level and volatility variations of the mortality rates.

\[\begin{array}{|c|c|c|}
\hline
\text{Decrease of experience longevity} & \text{FV}(0) & \text{FV(participation)} \\
\hline
\text{Case 1} & 306,124 & - \\
\text{Case 2} & 444,880 & 138,755 \\
\text{Case 3} & 368,278 & 62,154 \\
\text{Case 4} & 507,033 & 200,909 \\
\text{Case 5} & 489,089 & 182,965 \\
\hline
\end{array}\]

\[\begin{array}{|c|c|c|}
\hline
\text{Increase of experience longevity} & \text{FV}(0) & \text{FV(participation)} \\
\hline
\text{Case 1} & 323,548 & - \\
\text{Case 2} & 470,200 & 146,653 \\
\text{Case 3} & 378,090 & 54,543 \\
\text{Case 4} & 524,743 & 201,195 \\
\text{Case 5} & 501,145 & 177,597 \\
\hline
\end{array}\]

\(^1\) Note that this is a direct consequence of the general result giving the price of a European option under stochastic interest rates, cf. Brigo and Mercurio (2006)
### Table

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<th>Case</th>
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\[ \sigma_{\mu} = 0.25\% \]

\( \text{(Decrease of mortality volatility)} \)

We now give a few words about model calibration. First of all, the way such a model would be calibrated depends on its intended use. In the present case, the exercise is more academic and aims to understand the nature of the options contained in the different participation schemes. Beyond this, such a model, probably in its 2 factors version, might be used by a company in order to assess the profitability and viability of a specific participation scheme, in a given macro-economic and financial context. The fair value then represents the cost for the company of the insurance contract, given the existing financial and economic environment, and in particular given the possibilities in the market in terms of risk transfer and hedging.

All parameters of the interest rate model can be easily calibrated on existing interest rate options. Indeed, a huge liquid and deep market exists, so that nearly all types of interest rate risks - except very long ones (e.g. beyond 50 years) - can be transferred in the market. The model calibrated that way then leads to a "market consistent" price, reflecting in particular the cost of hedging or funding.

Concretely, calibration will be performed on the spot risk free yield curve deduced from a set of relevant linear products (government bonds or interbank products like interest rate swaps, deposit rates and futures), and a set of interest rate options quoted in the market (swaptions, caps).

The time-dependent mean reversion target \( \theta(t) \) is directly calibrated from the spot yield curve of the market at the reference date of valuation. Thanks to its time-dependency, perfect calibration is possible through an existing analytical formula, in function of other parameters \( a \) and \( \sigma_{\mu} \), so that whatever their values, calibration on the initial yield curve is ensured (see Brigo and Mercurio (2006) Chapter 3). Remark that the analytic formulae obtained in this paper already incorporates implicitly the link between the mean reversion target and the initial yield curve.
The second step consists to calibrate the remaining parameters on prices of existing quoted options (e.g. ATM swaptions).

For longevity risk, the context is completely different as the corresponding market is quasi non existing compared to interest rates. In particular, a market for quoted longevity options simply doesn't exist. The only existing instruments are linear products like longevity swaps or bonds.

The explicit formulae we obtain uses model ZC coupon prices (related to mortality and possibly also interest rates), so that again, calibration of the mean reversion target $\zeta(t)$ becomes actually the problem of selecting the appropriate ZC bonds prices. A possibility could be to consider (longevity) ZC bonds prices mortality rates deduced from existing longevity swaps or longevity bonds, in function of the existence of such products in the business environment of the company. Another one is to select an adequate experience mortality table, which is essentially what we did here.

The real issue is actually the calibration of other parameters, $b$ and $\sigma$. 

A possibility would be to calibrate these two parameters from historical data, like we would do in a real world version of the model, and by analogy to what the market participants did when valuing the first European options just after the introduction of Black-Scholes formula in 1973...

Another possibility would be to suppose a constant mean reversion target as well (Vasicek framework), and to calibrate the resulting three parameters either on historical data, or on existing longevity bonds or swaps, or simply on insurance premiums, by using similar technique as for interest rates (like proposed in Russo et al. (2011)).

Finally, calibration of the correlation coefficient, between interest rates and mortality rates, is even less evident. The value of 20% proposed in this paper was completely expert based: it relies on the assumptions that (1) an increase of mortality would lead to an increase of the offer of government bonds, leading to an increase of interest rates, and (2) the correlation level must be low.

8. CONCLUSION

In this article, we develop a modelling framework for the fair valuation of insurance contracts with various participation schemes, leading to tractable valuation formulae. Standard stochastic models for interest rates and mortality rates are assumed, the type of optionality contained in the different participation schemes are analysed and interpreted,
and simple analytical formulae for the fair value are obtained. In particular, the parallelism between financial and mortality participation is emphasized. Moreover, numerical results show that the fair valuation can vary a lot from a participation scheme to the other. We also see that even if we fix the guaranteed rate smaller than the market rate and use technical tables that are charged with respect to the experience, the fair value with participation may become greater than the premium at contract conclusion. This illustrates the fact that the level and the type of participation should always be fixed carefully by the company.

9. REFERENCES


MARGRABE, W., The value of an option to exchange one asset for another, *Journal of finance, 33*, 177-186 (1978)


10. APPENDIX: THE PURE ENDOWMENT NUMERAIRE

The pure endowment numeraire is defined by

\[ E_x(t,T) = \mathbb{E}_0 \left[ \exp \left( - \int_t^T (r(s) + \mu_x(s)) ds \right) \right] | \mathcal{F}_t \]

where:

- \( P \) is the price of a zero coupon bond
- \( \hat{p}_x \) is the adjusted survival probability.

These two processes are solution of the following stochastic differential equation:

\[
\begin{align*}
dP(t,s) &= r(t)P(t,s)dt - \sigma(t,s)P(t,s)dw_r(t) \\
d\hat{p}_x(t,s) &= \mu_x(t)\hat{p}_x(t,s)dt - \chi\hat{p}_x(t,s)(t,s)dw_\mu(t)
\end{align*}
\]

So we obtain for the new numeraire:

\[
d(E_x(t,T)) = P(t,T)d\hat{p}_x(t,T) + \hat{p}_x(t,T)dP(t,T)
\]

\[
= P(t,T)\hat{p}_x(t,T)\mu_x(t)dt - \chi(t,T)d\mu(t)
\]

\[
+ p_x(t,T)P(t,T)r(t)dt - \sigma(t,T)d\mu_r(t)
\]

\[
= E_x(t,T)r(t) + \mu_x(t)dt - E_x(t,T)\sigma(t,T)d\mu_r(t) + \chi(t,T)d\mu(t)
\]

\[
= E_x(t,T)(r(t) + \mu(t))dt - E_x(t,T)\Psi(t,T)d\mu_r(t)
\]

with: \( \Psi(t,T) = (\sigma^2(t,T) + \chi^2(t,T)) \frac{1}{2} \)

\[ dw_\mu(t) = \frac{\sigma(T,T)d\mu_r(t)}{\Psi(T,T)} + \frac{\chi(T,T)}{\Psi(T,T)} \]

If \( S \) is an asset we introduce a new underlying asset \( R \) defined by:

\[ R(t) = S(t)\exp \left( \int_0^t \mu_x(s) ds \right) \]

Taking into account the dynamics of \( S \) given by (7) the process \( R \) is solution of
the following stochastic differential equation:

\[ dR(t) = R(t)(r(t) + \mu_x(t))dt \]

\[ + R(t)\sigma_s(\rho dw_f(t) + \sqrt{1 - \rho^2} dw_s(t)) \]

\[ = R(t)r(t) + \mu_x(t)dt + R(t)\sigma_s dw_s(t) \]

with \( \langle dw_f(t), dw_E(t) \rangle = \rho^* \langle t, T \rangle dt = \frac{\sigma(t, T)}{\Psi(t, T)} \rho dt \)