FROM GAME THEORY TO SOLVENCY QUANTILE CALCULATION: CAPITAL ALLOCATION WITH USE IN NON-LIFE INSURANCE

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Abstract:

Capital allocation is an instrument for managing risk in an insurance company, especially by showing the diversification effect between lines of business. Based on a practical application of game theory, this article proposes a practical mapping between coherent measure of risk and coherent capital allocation. In terms of risk measure, VaR and TVaR have been selected, from which quantile-based allocation formulae can be derived. These formulae require the use of simulation, and it is shown here that the algorithm of Ruhm, Mango and Kreps (RMK) is especially adapted. Capital allocation has been applied in this article in two solvency calculation internal models. In this case, internal modelling of risks may change from one model to another. However, VaR shows more consistent results over time than TVaR. It is suggested that not only (axiomatic) coherent allocation, but also *consistent* allocation needs to be considered.

Keywords: Capital allocation; Aumann-Shapley; VaR; TVaR; Ruhm-Mango-Kreps algorithm; coherent capital allocation; coherent measure of risk; Solvency 2; internal model.

JEL Classification: C71, G11, G22, G31, G32

MSC Classification: 62P05, 65P05, 91B30

Résumé :

L'allocation du capital est un instrument de gestion du risque d'une compagnie d'assurance, en particulier parce qu'elle montre l'effet de diversification entre les lignes métiers. Fondé sur une application pratique de la théorie des jeux, cet article propose une correspondance entre la cohérence des mesures de risque et l'allocation cohérente de capital. En termes de mesures de risque, VaR et TVaR ont été sélectionnées, et les formules d'allocation en sont dérivées. Ces formules sous tendent l'utilisation de simulations, et on

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montre ici que l'algorithme de Ruhm, Mango et Kreps est adapté. L'allocation du capital a été appliquée dans cet article sur deux modèles internes de calcul de solvabilité. Dans ce cas, la modélisation interne d'un modèle à un autre peut changer. Cependant, la VaR montre des résultats dans le temps plus cohérents dans le temps que la TVaR. Ainsi, non seulement l'allocation cohérente (dans le sens axiomatique) mais aussi l'allocation cohérente *sur la durée* doit être considérée.

Mots-clés : Allocation du capital, Aumann-Shapley, VaR, TVaR, algorithme de Ruhm-Mango-Kreps, allocation de capital cohérente, mesure de risque cohérente, Solvabilité 2, modèle interne.

1. INTRODUCTION

Risk management is a concept that raise more and more interest in the financial industry. Not only in banking, in the wake of the financial crisis starting back in 2007-2008, but also in insurance, especially through the coming regulatory industry shake up in the European union, known as Solvency 2. One aspect of quantitative risk management we aim at dealing with in this paper is economic capital allocation. One aim of Solvency 2 is that economic capital calculation takes place by taking risk into account, unlike the current Solvency 1 regime. Computing economic capital, moreover on a risk-based approach is good. Managing risk by using the economic capital, by allocating it, is better. Some authors may not be totally convinced of the capital allocation approach, e.g. Gründl and Schmeiser (2007), Schradin and Zons (2003), Mango (2003), Venter (2002) or Venter (2007). However it constitutes the opportunity to embed risk management from the top to the risk taking people, i.e. the underwriters of the risk. Capital allocation is a field that has given rise to a relative extensive literature in the last 15 years, see references in Eling and Schmeiser (2010). This is still an active field of research, as there is no definitive consensus so as to choose a capital allocation method. We are not going to propose a new method. There are already a few, in addition to the one we are going to review, it is possible to mention proportional allocation, Myers-Read or covariance share methods. Reviews of methods are provided for instance by Scherpereel (2005), Kaye (2005), Balog (2010), Albrecht (2003), Albrecht and Koryciorz (2004), Venter (2004) or Cummins, Lin and Phillips (2006).

The aimed contribution of this paper is then threefold:

1. Coming back to one of the known background theory of capital allocation,

by using game theory. This provides an axiomatic theory for capital allocation;

2. Mapping this axiomatic approach of capital allocation with a well known axiomatic approach on risk measurement;
3. Showing how this theoretic and axiomatic approach can be used in a concrete setting.

As far as our knowledge goes, there are not that many examples of step by step allocation to be seen in the literature, and this is one of the aims of this article to contribute on this. Another aim is also to see what the changes of modelling from one internal model to another for the same company would bring. The most akin study one may find has been presented by McCrossan, Manley and Lavelle (2006), whereby the authors vary several parameters in order to see their effect on the allocations in an internal model approach. To be noted that reflecting the current flow of the literature, and partly because it is compulsory from Solvency 2, this is solely the capital allocation on an economic capital internal model that has been investigated here. An interesting contribution for using capital allocation with Solvency 2 standard formula can be found in Derien and Le Floc'h (2011).

This article is split as follows. Section 2 is dedicated to game theory background and risk measurement. Section 3 uses of this game theoretical background to derive an allocation principle. Section 4 and 5 provide capital allocation algorithms and illustrate them for a P&C UK insurer. This is followed by the conclusion.

2. PRELIMINARIES ON GAME THEORY AND RISK MEASURE

2.1 Game theory preliminaries

2.1.1 Cooperative and non-cooperative games

Game theory dedicates its focus on reviewing decision-making when conflict and/or cooperation situations occur, and is regularly found in economics, finance and actuarial science see e.g. Borch (1962), Lemaire (1985), Lemaire (1991), Pollack (2006). It is commonly admitted that game theory dates back to Cournot (1838), his contribution in economics is indeed analysed in Daughety (1988) and Touffut (2007).

As noted by Osborne and Rubinstein (1994), one taxonomy for game theory is to split it between noncooperative on the one hand and cooperative (or coalitional) games on the other hand, although they point out that this classification does not always ease the differences between the models. Leyton – Brown (2008a) gives the following heuristic definition of the noncooperative and the coalitional games:

- **Noncooperative game theory** is the *mathematical study of interaction between rational, self-interested agents"*. This is called noncooperative because "*individual is the basic modelling unit, and that individuals pursue their own interests"*.
- whereas "***cooperative/coalitional game theory** has teams as the central unit, rather than agents. (…) Given a set of agents, a coalitional game defines how well each group (or coalition) of agents can do for itself."*

For the topic we are interested in, we shall relate exclusively to coalitional game theory. It seems to be more frequent to meet the cooperative game theoretic approaches in the context of allocation problem, however noncooperative allocation is reviewed by Scherpereel (2005). As Kaye (2005) explains, risk allocation can be well described by coalitional game theory, as a sub-portfolio should benefit from being part of a larger component, i.e. a diversified portfolio. This is not a one-way relationship since the sub-portfolio in question is giving a diversifying benefit to the other sub-portfolios constituting the whole diversified portfolio. There could be a potential conflict so as to solve the challenge of fairly allocating the overall diversification benefit to each sub-portfolio, subcomponent of the larger diversified portfolio.

The two next paragraphs are dedicated to discrete and continuous noncooperative game theoretic settings, based on the contribution of Denault. Let us introduce the following notations:

$\mathcal{F}$: a firm, defined by a pair of variables (X, λ) defined below;

$\mathfrak{F}$: a set of firms, whereby $\mathcal{F} = (X, \lambda) \in \mathfrak{F}$;

$\mathcal{X}$: set of bounded random variables on the probability space [$\Omega, \mathcal{A}, \mathbb{P}$], whereby X $\in \mathcal{X}$, X being defined below;

$X = (X_1,...,X_n) \in \mathcal{X}^n$, where each X_i can be seen as the payoff per unit of an *i*th LOB, $i \in N = \{1,...,n\}$;

$\lambda = (\lambda_1,...,\lambda_n) \in \mathbb{R}^n$, where each λ_i can be seen as the number of units held of an *i*th LOB[1];

[1] Although λ notation is introduced now, its use will make sense from Section 3.2. Its use in Section 3.1.2 corresponds to the particular case of $\lambda_i = 1$.

ρ : risk measure, which formally maps $\mathcal{X}$ into $\mathbb{R}$;

$X(\lambda)=\sum_{i=1}^{n}\lambda_i X_i$: future period net worth, or payoff, of firm $\mathcal{F}$;

$\rho(X(\lambda))$: risk capital of the firm $\mathcal{F}$.

2.1.2 Discrete cooperative solution: the Shapley value

In game theory terminology, the term *discrete* is aka *atomic*. Next subsection aims at defining what it means. This enables then to derive the Shapley value.

Definition 2.1 (Coalitional game iro atomic game)

A coalitional game ($\mathcal{N}$,c) consists of :

- *i) a finite set $\mathcal{N}$ of n players, and*
- *ii) a cost (or characteristic) function c whereby a real number c($\mathcal{S}$) is associated to each subset $\mathcal{S}$ of $\mathcal{N}$, the former being called a coalition. $\mathcal{G}$ is denoted a set of games with n players.*

The Shapley value has given rise to a lot of interest, partly because of its axiomatic definition. Shapley first of all gives three axioms and then proves that a unique value (in the sense of Definition 3.2 below), satisfies these three axioms. In order to better understand axioms, we define the cost difference $\Delta_i(\mathcal{S})$, the interchangeability and the symmetry.

Definition 2.2

- The symbol $\Delta_i(\mathcal{S})$ represents the difference of two cost functions, namely $c(\mathcal{S}\cup\{i\})-c(\mathcal{S})$ for any set $\mathcal{S}\cup\mathcal{N},i\mathcal{S}$.
- Two players i and j are interchangeable in $(\mathcal{N},c)$, in the case each of them is giving the same contribution to each coalition $\mathcal{S}$, which includes neither i nor j, i.e. $\Delta_i(\mathcal{S})=\Delta_j(\mathcal{S})$ for all subsets $\mathcal{S}\subset\mathcal{N}$ with i, j $\mathcal{S}$.
- A player is a dummy if for each coalition $\mathcal{S}$ it gives the contribution $c(\{i\})$. That means $\Delta_i(\mathcal{S})=\rho(\{i\})$ for all $\mathcal{S}\subset\mathcal{N}$ with i $\mathcal{S}$.

From Definition 2.2, it is now possible to give the three axioms and the Shapley value Φ .

Axiom 2.1 (Symmetry) *If players i and j are interchangeable, then* $\Phi(\mathcal{N},c)_i=\Phi(\mathcal{N},c)_j$.

Axiom 2.2 (Dummy player) *For a dummy player* $\Phi(\mathcal{N},c)_i=c(\{i\})$.

Axiom 2.3 (Additivity over games) *Let us define the game* $(\mathcal{N},c_1+c_2)$ *whereby* $(c_1+c_2)(\mathcal{S})=c_1(\mathcal{S})+c_2(\mathcal{S})\forall\mathcal{S}\subseteq\mathcal{N}$.

For two games $(\mathcal{N},c_1)\text{and}(\mathcal{N},c_2),\Phi(\mathcal{N},c_1+c_2)=\Phi(\mathcal{N},c_1)+\Phi(\mathcal{N},c_2)$.

Remark 2.1 *Axiom 2.3 is ignored in respect of capital allocation.*

Just before the Shapley value, there is the following definition:

Definition 2.3 (Unique value) *The Shapley value is the only value satisfying Axioms 2.1, 2.2 and 2.3.*

Definition 2.3 enables the statement that this value exists and is unique. Moreover this value has an explicit form given in the next definition.

Definition 2.4 (Shapley value) *The Shapley value* $A^{\rho}_{Sh,i}$ *is defined as follows:*

$$A_i^{Sh,\rho} = \sum_{\mathcal{S}\in\mathcal{C}_i} \frac{(s-1)!(n-s)!}{n!}\big(c(\mathcal{S}) - c(\mathcal{S}\setminus\{i\})\big), i \in \mathcal{N} \qquad (1)$$

where $s = Card(\mathcal{S})$ and $\mathcal{C}_i$ represents all coalitions of $\mathcal{N}$ containing i.

A proof of Equation (1) can be found under Urban (2002), using material provided by Schlee (1999).

2.1.3 Continuous cooperative solution: the Aumann-Shapley value

The angle used by Denault (2001) is presented in this section, where it becomes clear that the Aumann-Shapley value constitutes a generalisation of the discrete (*atomic*) world to the continuous (*non-atomic*/*fractional*/*divisible*/*continuous*/ *non-discrete*) sphere. For a thorough review of this non-discrete approach and its roots in the theory of cooperative games with nonatomic players, the interested reader can refer to Urban (2002) or to the original source of Aumann and Shapley (1974).

The portfolio to which it is aimed to allocate capital are the **players**, the risk measure is represented by the **cost function**. The values are the **allocation principles**. In this continuous context, the presence of each portfolio can be scalable, i.e. a coalition can be built from *x*% of a portfolio A and *y*% of a portfolio B. In other words the portfolios must neither only be in nor only be out of a coalition, they can be both.

Aumann and Shapley use the interval [0,1] to represent the set of all players. The coalitions of fractional/non-atomic players are measurable subintervals, elements of a σ-algebra. Definitions 2.5 formalises the initial settings:

Definition 2.5 (Non-discrete cooperative game)

Theoric fractional players coalitional game (I, ℓ) is a measure set where I is a set and ℓ a σ-algebra of a measurable set of I. The components of I are the players and the components of ℓ are the coalitions. A non-discrete cooperative game consists of a function

$v:(I,\ell)\mapsto(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ with $v(\varnothing)=0$.

Set of all players When I=[0,1], the space of all functions is f(0)=0.

Non-atomicity A measure μ is non-discrete when for all $\mathcal{S}\in\ell,|\mu(\mathcal{S})|>0\exists T\subseteq\mathcal{S},T\in\ell\text{with}0<|\mu(\mathcal{S})|<|\mu(T)|.\mathcal{M}$ is the space of non-atomic/non-discrete measure.

Remark 2.2 *A non-discrete cooperative game is also known as a cooperative game with scalable players.*

The two first items of aforementioned Definition 2.5 clearly show the reference to the measure theory. A more intuitive presentation is introduced by Aubin (1979), Aubin (1981), Billera and Raanan (1981), Billera and Heath (1982) as well as Mirman and Tauman (1982). In order to make the framework of Definition 2.5 more understandable, some extra notations are needed. We need to introduce the vector $\Lambda\in\mathbb{R}_+^n$ for the full participation of each of the n players. To make Definition 2.5 operational, the set I is chosen so that $I=\{x\in\mathbb{R}^n/0\le x\le\Lambda\}$ and that ℓ is a Borel-σ-algebra. The cost function $r(\lambda)$ corresponds to $v([0,\lambda_1]\times\cdots\times[0,\lambda_n])$.

Definition 2.6 (Practical fractional players coalitional game) *A* coalitional game with scalable players (N, Λ ,r) consists of:

- a finite set N of players with $Card(N)$ = n,
- a positive vector $\Lambda\in\mathbb{R}_+^n$, which is the value of the full participation of each of the n players,
- a real-valued cost function r : $\mathbb{R}^n\to\mathbb{R},r:\lambda\mapsto r(\lambda)$, defined on $0\le\lambda\le\Lambda$.

For a portfolio of LOBs, Λ gives the actual length of each of the LOBs. For instance Λ can represent the premium volume of a LOB. Λ can also be expressed in a relative way, whereby the size of the LOB is expressed in a reference unit. Then a real cost function r is needed which is defined on $0\le\lambda\le\Lambda$. So the n-dimensional vector $\lambda\in\mathbb{R}_+^n$ is the so-called level of presence of each of n players in the coalition, and is a function defined on a continuous interval. Powers explains that as in Aumann-Shapley the portfolio needs to be *infinitely divisible*, the **level of presence** or participation in a portfolio of a member i is represented by λ, whereas the full portfolio or **full participation** is represented by Λ.

In the sequel, several theorems and definitions of the Aumann-Shapley value will be presented. The aim is to start with the most theoretical part to progress to the most practical definition of the Aumann-Shapley value. The original definition is as follows:

Theorem 2.1 (A first expression for the Aumann-Shapley value) *Let* (N,Λ,r) be a cooperative game with fractional players. Then there is a value[1]

$$\Phi_i^{AS}(N,\Lambda,r) = \int_0^1 \frac{\partial r}{\partial \lambda_i}(t\Lambda)dt,$$

which is designated as the **Aumann-Shapley value**.

It means that the costs per component for the *i*th portfolio are the average of the marginal costs, where the volume of all portfolios grows linearly from 0 to Λ.

Proof *See e.g. Urban (2002).*

If a homogeneous risk measure is used, the calculation of the Aumann-Shapley value can be computed in an easier manner, thanks to the following lemma of mathematic analysis (see e.g. Simon and Blume (2008)).

Lemma 2.1 (k-homogeneous function) *Let* $r:\mathbb{R}^n \to \mathbb{R}, \lambda \mapsto r(\lambda)$ *be a* k*-homogeneous function, i.e. for* $\gamma \in \mathbb{R}$ *there is* $r(\gamma\lambda) = \gamma^k r(\lambda)$, *then* $\frac{\partial r(\lambda)}{\partial \lambda_i}$ *is a* $(k-1)$ *homogeneous function.*

Proof *Assume* $\Lambda \in \mathbb{R}^n$ *and* $r_\gamma(\lambda) = r(\gamma\lambda)$. *Then the following calculations hold:*

$$\frac{\partial r_\gamma}{\lambda_i}(\Lambda) = \gamma \frac{\partial r}{\partial \lambda_i}(\gamma\Lambda) \text{and} \frac{\partial r_\gamma}{\partial \lambda_i}(\Lambda) = \frac{\partial \gamma^k r}{\partial \lambda_i}(\Lambda) = \gamma^k \frac{\partial r}{\partial \lambda_i}(\Lambda).$$

From that follows:

$$\gamma^{k-1}\frac{\partial r}{\partial \lambda_i}(\Lambda) = \frac{\partial r}{\partial \lambda_i}(\gamma\Lambda).$$

Theorem 2.2 (The Aumann-Shapley value for k-homogeneous function)

Assume $(X_1,\ldots,X_n)'$ a vector of random variables and ρ a partially differentiable homogeneous risk measure. Moreover (N,Λ,r) is a cooperative game with fractional players with a cost function $r(\lambda) = \rho(\sum_{i=1}^n \lambda_i X_i)$. The Aumann-Shapley value can be computed as follows:

$$\Phi_i^{AS}(N,\Lambda,r) = a_i^{AS}(X,\lambda) = \frac{\partial r}{\partial \lambda_i}(\Lambda).$$

Proof *From Lemma 2.1 and as* r *is homogeneous, there is:*

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[1] This expression implies that $\lambda_i = t\Lambda_i$, where $t \in (0,1]$ stands for the participation or involvement proportion (terminology used by Powers) expressed by $t = \frac{\lambda_i}{\Lambda_i}$.

$$\Phi_i^{AS}(N,\Lambda,r)=\int_0^1\frac{\partial r}{\partial\lambda_i}(t\Lambda)dt=\int_0^1\frac{\partial r}{\partial\lambda_i}(\Lambda)dt=\frac{\partial r}{\partial\lambda_i}(\Lambda).$$

2.2 Risk measures

2.2.1 Definitions

The following definition may be proposed:

Definition 2.7 (Economic capital) *According to de Weert (2011), economic capital is the amount of capital that a financial institution needs to hold to cover the risks it is facing. This represents the amount of money that is needed to secure survival in a severely adverse scenario. This means that, if the available capital of a financial institution exceeds its economic capital, the financial institution is able to weather heavy shocks.*

Economic capital is generally valued through the computation of a risk measure. The following definition from Sandström (2011), Denuit, Dhaene, Goovaerts and Kaas (2005) or to some extent Cadoux, Loizeau and Partrat (2003) is given here:

Definition 2.8 (Risk measure) *A risk measure is a functional ρ mapping a risk X to a nonnegative number $\rho(X)$, which can be infinite. It represents the amount of extra cash added to X in order to make the risk* acceptable.

Referring to Definition 2.6, let us show the link between the risk measure and the cost function, which has been only alluded to before. Indeed the cost function can be identified with a risk measure through:

$$r(\lambda)=\rho\left(\sum_{i=1}^{n}\lambda_i X_i\right).$$

Ultimately, Sandström provides a general link between Definition 2.8 and the one for economic capital in following terms:

Definition 2.9 (Economic capital function of a risk measure ρ) *Economic capital can be defined through the following equation:*

$$\text{EconomicCapital}=\kappa\rho(X),\forall\kappa>0. \tag{2}$$

This definition is actually a consequence from the concept of solvency. Indeed insolvency is avoided, assuming a specified time horizon and given a tolerance level $1-\alpha$, if the insurer holds at least $\rho(X)$. This is the assumption that in $(1-\alpha)$ % of the cases, economic capital will be enough to avoid insolvency. $1-\alpha$ is alaso known as confidence level. $\rho(X)$ is valued via a risk measure.

2.2.2 *Coherent risk measure*

There is a particular set of axioms known as *coherent measure of risk* (see Artzner, Delbaen, Eber, and Heath (1999) and Artzner (1999)), intended to be desirable for characterising a risk measure as *good* to effectively regulate or manage risks.

Definition 2.10 (Coherent measure of risk) *A risk measure satisfying the following four axioms is called coherent:*

- ***Positive homogeneity**: The risk of a multiple is the multiple of this risk:*
$$\rho(\kappa X) = \kappa\rho(X), \forall \kappa \geq 0.$$
- ***Subaddivity:** Pooling together risk adds up to less than the sum of these risks taken one by one:*
$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$
- ***Translation invariance**: Adding a constant to a risky position increases the risk by the amount of this constant:*
$$\rho(X+\nu) = \rho(X) + \nu, \forall \nu \in \mathbb{R}.$$
- ***(Weak) monotonicity:** Capital for a less risky position X than a riskier position Y is smaller for X than for Y:*
$$X \leq Y \rho(X) \leq \rho(Y).$$

Let us explain in more depth positive homogeneity and subadditivity:

Positive homogeneity is often required for technical reasons, e.g. in the case of a quota share reinsurance treaty. Koryciorz (2004) makes a very important note stating that this is often misinterpreted in the insurance context. While this is seen in finance as a simple proportional multiplier of the volume of a portfolio[1], this interpretation is transposable to the insurance context only insofar as one deals with perfectly positively dependent risks, see e.g. Mildenhall (2004) or Mildenhall (2006). This has not to be confused with homogeneous risks in the insurance context, assumed not to be perfectly positively dependent and having the characteristic to diversify by virtue of the losses mutualisation.

[1] So if $\kappa = 2$, the volume of the portfolio is multiplied by two.

Subadditivity: This property puts in formal terms the pooling of risks in the insurance industry. It allows for diversification effect (see e.g. Long and Whitworth (2004)), defined as follows:

$$\text{Diversification} = \sum_{i=1}^{n} \rho(X_i) - \rho\left(\sum_{i=1}^{n} X_i\right).$$

whereby Diversification ≥ 0 for subadditive risk measures.

It can be noted that Tsanakas (2009) has tried to use *convex risk measures* in the context of capital allocation, but as the author concludes himself, further research is needed to make this explored framework more realistic.

3. ALLOCATION PROBLEM USING GAME THEORY

Definition 3.1 (Allocation principle) *Let the set of firms* $\mathfrak{F}$ *be defined as a set of pairs* (X, λ), *where* $X \in \mathcal{X}^n$ *and* $n \in \mathbb{N}_+^*$. *Let* ρ *be a given risk measure. An allocation principle is a function* $A^\rho : \mathfrak{F} \to \mathbb{R}^n$ *mapping each allocation problem (X,* λ *) into a unique allocation on* $\mathfrak{F}$ *:*

$$A^\rho : (X, \lambda) \mapsto \begin{bmatrix} A_1^\rho(X, \lambda) \\ A_2^\rho(X, \lambda) \\ . \\ . \\ . \\ A_n^\rho(X, \lambda) \end{bmatrix} \text{where} \sum_{i=1}^{n} A_i^\rho(X, \lambda) = \rho(X(\lambda)). \tag{3}$$

$A_i^\rho(X, \lambda)/\lambda_i$ is called the per-unit risk contribution of position i.

3.1 In the discrete case

3.1.1 Atomic games

Definition 3.2 (Value of an atomic game) *A value is a function* $\Phi : \mathcal{G} \to \mathbb{R}^n$ *that maps each game (* $\mathcal{N}$ *,c) into a unique allocation:*

$$\Phi:(\mathcal{N},c)\mapsto\begin{bmatrix}\Phi_1(\mathcal{N},c)\\ \Phi_2(\mathcal{N},c)\\ .\\ .\\ .\\ \Phi_n(\mathcal{N},c)\end{bmatrix}=\begin{bmatrix}A_1^{\rho}(X,\lambda)\\ A_2^{\rho}(X,\lambda)\\ .\\ .\\ .\\ A_n^{\rho}(X,\lambda)\end{bmatrix}\text{where}\sum_{i\in\mathcal{N}}A_i^{\rho}(X,\lambda)=c(\mathcal{N}). \tag{4}$$

It is straightforward to identify that Equation (4) is the same, *mutatis mutandis*, as Equation (3).

Let us explain the rationale of the two preceding definitions. Game theory studies situations where players adopt various strategies in order to reach in an optimal way their own goal. Further to Definition 2.1, the aim of each atomic player is to minimise their cost, measured by the cost function. Therefore their strategy is either to join or not to join a coalition. Henceforth one of the main questions for each participating part to the coalition is the amount allocated to each part, here atoms. This is actually formalised by the concept of value, as defined is Definition 3.2.

Then, the next definition is crucial, introducing the idea of the core. If a player ends up getting allocated a share $A_i^{\rho}(X,\lambda)$ superior to its own cost $c\{i\}$, this player will rationally threaten to leave the coalition. More generally, if from coalitions $\mathcal{S}\subseteq\mathcal{N}$ that $\sum_{i\in\mathcal{S}}A_i^{\rho}$ is more than $c(\mathcal{S})$ then every player i in $\mathcal{S}$ may carry more allocated $A_i^{\rho}(X,\lambda)$ than its cost function, i.e. carrying more economic capital than it has to pay to bear it. The core is then defined so that this kind of threat does not happen, neither for an isolated player i nor for a coalition.

Definition 3.3 (Core of a game) *The core of a coalitional game* $(\mathcal{N},c)$ *is the set of allocation* $A^{\rho}\in\mathbb{R}^n$ *whereby* $\sum_{i\in\mathcal{S}}A_i^{\rho}\le c(\mathcal{S})$ *for all coalitions* $\mathcal{S}\subseteq\mathcal{N}$.

The aim is then to find an allocation which satisfies the core, as it becomes clear from the explanation above that it is regarded as a *good* or *desired* allocation. Charpentier (2007) gives an expression of that by saying that the allocated capital has to be in that case lower than the stand-alone risk itself. Urban (2002), in the context of cooperative games, cites the three following solutions satisfying the core: the von Neumann-Morgernstern solution, the nucleolus and the Shapley value.

The first two methods are reviewed in Kaneko and Wooders (2004) and Urban (2002). Scherpereel (2005) and Mandl (2005) also provide a description of the nucleolus.

The Shapley value, more workable than the two others is used in the next subsection. Indeed Urban stresses that one drawback of the von Neumann-Morgenstern solution is that there is either no imputation or an infinity of them. Moreover the interpretation of the nucleolus concept with its *dissatisfaction function* can be challenging.

3.1.2 Discrete (or atomic) case: the Shapley value

The allocation formula itself is derived from Definition 2.4 and is presented below:

Definition 3.4 (Allocation as per the Shapley value) *Let us assume a vector of random variables* $(X_1,\ldots,X_n)'$ *and a risk measure* ρ*. The allocation coefficients of the allocation according to Shapley using the risk measure* ρ *is then expressed by:*

$$\mathcal{A}_i^{Sh,\rho}(X)=\frac{\left[\sum_{\mathcal{S}\in\mathcal{C}_i}\frac{(s-1)!(n-s)!}{n!}\left(\rho(\sum_{j\in\mathcal{S}}X_j)-\rho(\sum_{j\in\mathcal{S}\setminus\{i\}}X_j)\right)\right]}{\rho(\sum_{j\in\mathcal{N}}X_j)},i\in\mathcal{N}.$$

The intuition behind the Shapley value can be found under Scherpereel (2005), who uses material from Hamlen, Hamlen and Tschirhart (1977). Leyton-Brown (2008b) gives a useful insight as well. The idea is that a firm (the whole coalition) is built by a sequential addition of divisions (players). Each division joining the firm will get charged with the necessary risk capital it is adding, i.e. $\left(c(\mathcal{S})-c(\mathcal{S}\setminus\{i\})\right)$. The key is that there is no particular order at which each division is joining the firm, each entry order is assumed equiprobable. So when a set of divisions (coalition) is fixed, the likelihood for the entry of a new division $i\in\mathcal{S}$ as $\mathcal{S}$th division in the firm is expressed by the factor $\frac{(s-1)!(n-s)!}{n!}$.

3.2 Continuous case: the Aumann-Shapley value

Let us define, in the same way as in Definition 3.2, *mutatis mutandis*, the value in the context of a non-atomic game.

Definition 3.5 (Fuzzy value) *A (fuzzy) value is a function* Φ *mapping a coalitional game* (N,Λ,r) *to a unique per-unit allocation vector* $a\in\mathbb{R}^n$*, i.e.* $\Phi(N,\Lambda,r)=a$*, such that* $\Lambda^T a=r(\Lambda)$.

The latter can be written as $\sum_{i=1}^{n}\Lambda_i a_i(X,\lambda)=r(\Lambda)$, where the dependence of a on (X,λ) is stressed.

It is referred here to the fuzzy set theory of Zadeh (1965). The fuzzy sets are sets of which the components have *degrees* of membership. This is an extension of the notion of set, whereby the membership of an element is assessed on a binary basis, either the element is in or out of the set. The fuzzy set notion enables a gradual membership of the element in a set. This generalises the classical set, which takes indeed the values 0 or 1 only.

As Denault notes, Definition 3.5 gives an allocation principle, that can be generalised when we make use of Theorem 2.1 for scalable players. Following Theorem 2.2, it is possible to define the capital allocation using Aumann-Shapley method.

Definition 3.6 (Allocation coefficients according to Aumann-Shapley) *Let a vector of random variables* $(X_1(\lambda_1),\ldots,X_n(\lambda_n))'$ *be the payoff of each LOB, whereby* $\Lambda=(\Lambda_i,\ldots,\Lambda_n)'$ *is the length of each LOB, and* ρ *is a risk measure that is partially differentiable by* λ_i for $i=1,\ldots,n$.

Let $r(\lambda)=\rho(\sum_{i=1}^{n}X_i(\lambda_i))$. Then the allocation coefficients of the allocation according to Aumann-Shapley using the risk measure ρ is defined as:

$$\mathcal{A}_i^{AS,\rho}=\frac{\left(\Lambda_i\int_0^1\frac{\partial r}{\partial\lambda_i}(t\Lambda)dt\right)}{r(\Lambda)}\text{for}\, i=1,\ldots,n.$$

When ρ is a homogeneous risk measure and $r(\lambda)=\rho(\sum_{i=1}^{n}\lambda_i X_i)$ then

$$\mathcal{A}_i^{AS,\rho}=\frac{\left(\frac{\partial r(\Lambda)}{\partial\lambda_i}\Lambda_i\right)}{r(\Lambda)}\text{for}\, i=1,\ldots,n.$$

4. COHERENT CAPITAL ALLOCATION

4.1 Coherent allocation

4.1.1 Definition of coherent allocation

The seminal paper of Denault (2001) takes an axiomatic approach to the capital allocation problem. Three other capital allocation axiomatics can be enumerated, which have not been extensively reviewed in the literature as opposed to the one introduced by Denault: Kalkbrener (2005), Kim (2007) and Gourieroux and Monfort (2011). It has been chosen to stick to the one of Denault, aiming at the outcomes of Section 4.2. The idea is of course to allocate economic capital, the latter being reflected by a risk measure. The viewpoint is here to have desirable properties for having a *reasonable* risk measure. The axiomatic translation of reasonable is here coherent. The starting point is therefore the (axiomatic) coherent risk measure principle presented above in Definition 2.10. Denault assumes in his paper all risk measures to be coherent. We will therefore stick to this assumption for the rest of this section.

It is aimed to obtain a *reasonable* risk capital allocation. The same idea as for a reasonable risk measure applies to a reasonable risk capital allocation. Denault introduces in his article three axioms necessary for a capital allocation to be coherent. We use below the definitions of Buch and Dorfleitner (2008).

Definition 4.1 (Coherent allocation principle) *An allocation principle* A^{ρ} *on* $\mathfrak{F}$ *is coherent if the following properties hold for all* $\mathcal{F} = (X, \lambda) \in \mathfrak{F}$ *:*

- **No undercut:**

$$\forall M \subseteq N, \sum_{i \in M} A_i^{\rho}(X, \lambda) \leq \rho\left(\sum_{i \in M} \lambda_i X_i\right).$$

- **Symmetry:** *Joining the subsets* $M \subseteq N$ *and* $\{i, j\}$, *if portfolios* i *and* j *are interchangeable, then they both make the same contribution, i.e.*

$$\rho\left(\sum_{k \in M} \lambda_k X_k + \lambda_i X_i\right) = \rho\left(\sum_{k \in M} \lambda_k X_k + \lambda_j X_j\right) A_i^{\rho}(X, \lambda) = A_j^{\rho}(X, \lambda).$$

- **Riskless allocation:** *If* X_n *is a riskless instrument worth 1, then*

$$A_n^{\rho}(X, \lambda) = \rho(\lambda_n) = -\lambda_n.$$

Let us explain the rationale behind each of the sub-axioms given in the above definition as per Denault.

The **no-undercut** property ensures that a portfolio cannot get more capital allocated as it would have if this portfolio were a separate entity. This property ties up actually with the subadditivity axiom of a risk measure introduced above. If a portfolio is joining the company, it cannot be allocated more risk capital than what it could have brought to this company. In other words the total risk capital of the company should not increase more than the new entrant's portfolio own risk capital. The justification of this property is twofold, the next two arguments being intimately linked:

- Firstly it is consistent with the subadditivity axiom, reflecting the **diversification benefit** of merged portfolios (“a merger does not create extra risk” Artzner et al. (1999));
- Secondly the whole rationale of capital allocation is aiming at a fair allocation of risk capital, and the no-undercut property ensures that the

fairness rationale is respected.

The **symmetry** principle is there to ensure that the allocation to one portfolio depends exclusively on its risk contribution to the company.

The **riskless allocation** axiom tells that a portfolio without risk should be allocated exactly its risk measure. This risk measure is negative, therefore all other things being equal, when one portfolio sees its cash position increasing, its allocated capital decreases by exactly the same amount.

Denault introduces a fourth axiom of non-negativity:

Definition 4.2 (Non-negativity of the allocation) *A coherent allocation is non-negative if* $A_i^{\rho}(X,\lambda) \geq 0, \forall i \in N$.

It means that an allocation satisfying Definitions 4.1 and 4.2 is a so called non-negative coherent allocation of risk capital. Denault requires this axiom, which is traditionally overviewed in the materials using the axiomatic he introduced, because it may lead otherwise to some difficulties from a practical point of view for Risk Adjusted Performance Metric (in short, a risk adjusted return ratio) (RAPM) (see e.g. Doff (2007)) computation, where return may be divided by negative allocated capital, quantity which would not be straightforward to interpret.

4.1.2 Application to the discrete case, relating to the Shapley value

The link between the Shapley value and the axiomatic approach developed under Definition 4.1 is the following. Two of the three coherence axioms have been already gathered, namely symmetry (Axiom 2.1) and riskless allocation, the latter corresponding to the dummy player (Axiom 2.2). There is one coherent allocation axiom missing, which is (strong) subadditivity. Therefore a custom assumption is to consider the cost function to be strongly subadditive, i.e.:

$$c(\mathcal{S}_1 \cup \mathcal{S}_2) \leq c(\mathcal{S}_1) + c(\mathcal{S}_2).$$

This assumption is used in the following definition:

Definition 4.3 (Strongly subadditive game) *A coalitional game is strongly subadditive if based on a strongly subbadditive cost function:*

$$c(\mathcal{S}_1 \cup \mathcal{S}_2) + c(\mathcal{S}_1 \cap \mathcal{S}_2) \leq c(\mathcal{S}_1) + c(\mathcal{S}_2) \forall \mathcal{S}_1 \subseteq \mathcal{N} \text{and } \mathcal{S}_2 \subseteq \mathcal{N}.$$

What leads to the needed strong subadditivity theorem:

Theorem 4.1 (The Shapley value and the core) *If a game* $(\mathcal{N}, c)$ *is strongly subadditive, the Shapley value is in the core.*

Proof *See Shapley (1971-1972).*

As explains Scherpereel (2005), the core-compatibility of the Shapley value as per Theorem 4.1 is equivalent to the no-undercut of the coherent allocation of Definition 4.1. As a result, the Shapley value, under the condition of Theorem 4.1 represents the value of a coherent allocation. However, there is an important caveat, coming from Theorem 4.1, in the form of Theorem 4.2.

Theorem 4.2 (Linearity of the risk measure) *Let us assume ρ a positively homogeneous risk measure, so that $\rho(0)=0$. Assuming c to be defined over the set of subsets of random variables in L^∞, implies:*

$$c(\mathcal{S}) \equiv \rho\left(\sum_{i\in\mathcal{S}} X_i\right).$$

Then when c is strongly subadditive, ρ is linear.

Proof *See Denault (2001).*

The assumption of homogeneity of the risk measure is necessary if it is aimed to stay in the framework of consistency of the risk measure and the capital allocation. Indeed Definition 2.10 is clearly imposing a positively homogeneous measure of risk. And because of Theorem 4.2, it comes out that the cost function (game theory viewpoint) is strongly subbadditive i.e. the risk measure (capital allocation viewpoint) is linear, as Denault proves in his paper. As a result, the risk is fully additive, whereas it would have been preferred to have a subadditive one. This is a very stringent consequence, because a linear risk measure means that there is no possibility for a diversification effect. Denault concludes that a convincing proof of coherent allocation falls short. He proposes however to stick to the game theoretic approach using a slightly changed definition of a coherent allocation, as exposed under Section 4.1.3.

Before moving to the next section, the Shapley value calculation is explicitly reviewed in the literature, some saying even that it is one with a widespread use (e.g. Zhang (2008)). However, on top of the theoretical drawbacks of the Shapley value, a practical one is repeatedly found in the literature (e.g. Urban (2002), Balog (2010), Scherpereel (2005), D'Arcy (2011) or Diers (2007)). It is argued that for more than three players/coalitions, which is a fairly small number when one thinks about the numbers of LOBs in an insurance company, the computation can become quite complicated. Namely a company with n LOBs ends up with $2^n - 1$ possible coalitions. Moreover Scherpereel explains that the complicated computation of the Shapley value leads the latter generally not to be in the core, which in turn is a major drawback from an axiomatic argument point of view. Shapley turns around this critical point by using convex games, which are core compatible. However capital allocation is not always convex, otherwise the underlying risk measure should be comonotonic (cf. Delbaen (2002)), which is not a condition that is always given.

In order to finish up on the drawbacks, Kaye (2005) mentions that the fact that players have to be atomic can constitute a problem. He gives a very simple example taking three LOBs A, B and C. If A is split into two parts, the Shapley has to be recomputed. By doing that, the allocation to B and C will change, what may seem odd as a property everything else being taken constant.

4.1.3 Application to the continuous case relating to the Aumann-Shapley value

The definition of the coherence of fuzzy values is introduced under Definition 4.4. This represents the technical condition for the coherence of capital allocation in coalitional games with scalable players.

Definition 4.4 (Coherence of fuzzy values) *Let r be a coherent risk measure. A fuzzy value $\Phi:(N,\Lambda,r)\mapsto a\in\mathbb{R}^n$ is coherent if:*

- *Φ satisfies the five properties below and,*
- *a is an element of the fuzzy core.*

The five properties are indeed:

- **Aggregation invariance** *property: assumes the risk measures r and $\bar{r}$ satisfy $r(\lambda)=\bar{r}(\Gamma\lambda)$ for some $m\times n$ matrix Λ and every λ so that $0\le\lambda\le\Lambda$ then*
$$\Phi(N,\Lambda,r)=\Gamma^T\Phi(N,\Gamma\Lambda,\bar{r}).$$
- **Continuity** *property: the mapping Φ is continuous over the normalised vector space M^n of continuously differentiable functions $r:\mathbb{R}_+^n\to\mathbb{R}$ vanishing at the origin.*
- **Monotonicity** *property (aka non-negativity under r non-decreasing): If r is non-decreasing, i.e. $r(\lambda)\le r(\lambda^*)$ whenever $0\le\lambda\le\lambda^*\le\Lambda$, then*
$$\Phi(N,\Lambda,r)\ge 0.$$
- **Dummy player allocation** *property: if i is a dummy player, i.e.*
$$r(\lambda)-r(\lambda^*)=(\lambda_i-\lambda_i^*)\frac{\rho(X_i)}{\Lambda_i}$$

whenever $0\le\lambda\le\Lambda$ and $\lambda_j=\lambda_j^*$ for $j\ne i$, then
$$a_i=\frac{\rho(X_i)}{\Lambda_i}.$$

- **Fuzzy core** *property: the allocation principle* $\Phi(N,\Lambda,r)$ *is in the fuzzy core of the game* (N,Λ,r) *if for every* λ *,* $0 \le \lambda \le \Lambda$ *,*

$$\lambda^T \Phi(N,\Lambda,r) \le r(\lambda)$$

or

$$\Lambda^T \Phi(N,\Lambda,r) = r(\Lambda).$$

The idea linked to Definition 4.4 is to prepare the path towards a coherent allocation principle, as disclosed in Table 1.

Coherent fuzzy value	**Coherent allocation of risk capital**
aggregation invariance	*symmetry*
continuity	N/A
monotonicity	N/A
dummy player allocation	riskless allocation
fuzzy core	no undercut
not in Definition 4.4	full allocation

Table 1: Mapping between coherent fuzzy values and coherent allocation of risk capital

Interestingly, the first three properties of Definition 4.4 are not especially in line with the principle of coherent capital allocation. In the bottom part of Table 1, two necessary properties are provided by the definition above useful for a coherent allocation, whereas a last one is still missing but will be introduced soon in the sequel; taken one by one:

- ***Aggregation invariance*** is like the symmetry property, because equivalent risks should be allocated the same amount of capital. However Buch and Dorfleitner (2008) as well as Balog (2010) show that symmetry, although an axiom, is not a desirable property. This is because assuming a linear risk measure is equivalent not to account for diversification, where capital allocation aims at the opposite: to account for diversification.
- ***Continuity***, which does not correspond to a specific property for capital allocation coherence, but is nevertheless desirable for ensuring that similar risk measures give similar allocations.
- ***Monotonicity***, which is not required for a coherent capital allocation either, but is simply there to satisfy the requirement that the more risky a business is, the more it receives capital (See for instance Rafels (2006)).

- The ***dummy player*** property represents the riskless portfolio, whereby a LOB without risk (e.g. constituted only of cash) should attract a negative allocation. This position may seem theoretic, but is there to ensure that when the cash amount arising from a LOB increases, its capital allocation should decrease accordingly.
- The ***fuzzy core*** property, obtained from Definition 3.5 allows neither undercut from any player nor coalition with fractional players. This is a critical property for enabling a *fair allocation*.
- The ***full allocation principle*** is not embedded in any of the properties of Definition 4.4. To make the allocation to non-atomic players comprehensive, this is indeed the Aumann-Shapley value which fills this necessary gap as it will be seen below.

4.2 Links between coherent risk measure and coherent allocation

From Definition 3.6, using the notion of a homogeneous function, it is now possible to complete Table 1 satisfactorily under the form of Table 2.

Coherent risk measure	Coherent allocation of risk capital
positive homogeneity	full allocation
subadditivity	no undercut
translation invariance	riskless allocation
linearity	*symmetry*
monotonicity	*N/A*

Table 2: Mapping between coherent risk measures and coherent allocation of risk capital

The idea behind this table is the one of Denault, i.e. offering a consistent axiomatic approach for capital allocation. Buch and Dorfleitner (2008) explicitly show this relationship, which is rather underlying in the article of Denault. Now that coherent risk measurement and coherent allocation are mapped, let us apply the Aumann-Shapley value to two well known risk measures.

4.3 Application using tail-based measures

As seen in Definition 3.6, Aumann-Shapley allocation principles applied to our context require a differentiable risk measure. Henceforth the aim of this section is to review the differentiability of the two well known risk measures, which can be used in economic capital calculation, Value at Risk (VaR) and Tail Value at Risk (TVaR). It can be interesting to note that in relation to a broader notion of differentiability than the one presented here, Fischer (2003) or Fischer (2004) has defined weakened differentiability properties. It allows him to use an alternative to the quantile-based risk measures exposed here, focusing on one-sided moments, and even to work not only on continuous distributions but also discrete ones. His findings are developed in the same framework as just explained in the preceding Section 3.2.

4.3.1 VaR and capital allocation

In this subsection our aim is to compute the risk contribution associated with VaR via differentiation. However Tasche1 points out that the quantile function $\mathrm{VaR}_{\alpha}(Z_{\lambda})$ is generally not differentiable in λ. This is why some technical assumptions on the joint distribution of the vector $(X_1,\ldots,X_n)'$ have to be taken. Among them there is one which really matters: at least one among the random variables X_i has to have a continuous distribution density.

Assumption 4.1 *Let us take the vector of random variables* $(X_1,\ldots,X_n)$ *where* $n \geq 2$ *and some constant* $t > 0$,

$$F : \mathbb{R}\times\mathbb{R}^{n-1} \to [0,\infty),(t,x_2,\ldots,x_n)$$

is the density of the conditional distribution of X_1 *given* $(X_2,\ldots,X_n)'$ *and* $\alpha \in (0,1)$. *Then* $(X_1,\ldots,X_n)'$ *satisfies the following four conditions:*

- *For fixed* $x_2,\ldots,x_n$ *the function* $t \mapsto F(t,x_2,\ldots,x_n)$ *is continuous in t.*
- *The expression*

$$\mathbb{R}\times\mathbb{R}\setminus\{0\}\times\mathbb{R}^{n-1} \to [0,\infty),(t,\lambda) \mapsto E\left[F\left(\lambda_1^{-1}\left(t-\sum_{j=2}^{n}\lambda_j X_j\right),X_2,\ldots,X_n\right)\right]$$

has finite values and is continuous.

- *For all* $\lambda \in \mathbb{R}\setminus\{0\}\times\mathbb{R}^{n-1}$ *there is*

$$0 < E\left[F\left(\lambda_1^{-1}\left(Q_{\alpha}(\lambda)-\sum_{j=2}^{n}\lambda_j X_j\right),X_2,\ldots,X_n\right)\right].$$

- *For all* $i = 2,\ldots,n$ *the expression*

$$\mathbb{R}\times\mathbb{R}\setminus\{0\}\times\mathbb{R}^{n-1} \to \mathbb{R},(t,\lambda) \mapsto E\left[X_i F\left(\lambda_1^{-1}\left(t-\sum_{j=2}^{n}\lambda_j X_j\right),X_2,\ldots,X_n\right)\right]$$

has finite values and is continuous.

Assumption 4.1 then given, it is now possible to give the formal expression of the VaR's partial derivative.

Lemma 4.1 (Partial derivative of) *Let a vector of random variables* $(X_1,\dots,X_n)$ *satisfy the Assumption 4.1 and* $\alpha \in (0,1)$. *In addition let* $U = \mathbb{R}\setminus\{0\}\times\mathbb{R}^{n-1}$ *and* $Z_\lambda = \sum_{i=1}^{n}\lambda_i X_i$ *for* $\lambda \in U$ *. Then with* $VaR_\alpha(Z_\lambda): U \to \mathbb{R}$ *and*

$$VaR_\alpha(Z_\lambda) = -\inf\{z \in \mathbb{R} \mid \mathbb{P}[Z_\lambda \le z] \ge \alpha\} \quad \text{for} \quad \lambda \in U$$

$$\frac{\partial VaR_\alpha(Z_\lambda)}{\partial \lambda_i}(\lambda) = E[X_i \mid Z_\lambda = VaR_\alpha(Z_\lambda)] \quad \text{for} \quad i = 1,\dots,n. \tag{5}$$

Proof *See Tasche (2000).*

Another representation is Jorion who calls it *Marginal VaR* and shows a relation to the Beta and hence the CAPM. Historically Equation (5) is presented by Hallerbach where the differentiability of $\mathrm{VaR}_\alpha(Z_\lambda)$ is not reviewed. This is different to Lemma 4.1 where it is specifically addressed. Moreover, Tasche points out that in the event that Assumption 4.1 does not hold, Equation (5) used as a support for a risk contribution computation might still have a good chance to remain suitable to the risk measure. Gourieroux have also presented Equation (5) with a joint density for the vector $(X_1,\dots,X_n)'$.

4.3.2 On the non-coherence of VaR

VaR is often seen as not coherent (see e.g. Scherpereel (2005), Koryciorz (2004) or Haugh (2010)), i.e. violating the axioms seen under Definition 2.10 as introduced by Artzner2. This is because VaR is not deemed to respect the subadditivity axiom. That is to say $\mathrm{VaR}(X+Y)$ might be higher than $\mathrm{VaR}(X)$ + $\mathrm{VaR}(Y)$, which violates the principle asserted by Artzner et al. (1999) that “a merger does not create extra risk”. In other words, the merged risks X and Y should lead to risk mitigation from these risks added together, also named as *risk diversification*. Artzner et al. do see here a “natural requirement”, although not recognised by some authors who argue subadditivity is nothing more than a nice mathematical property because not observable in reality, see for instance Rootzén and Klüpperberg (1999). Moreover Heyde, Kou and Peng (2009) do not see any contradiction between VaR and subadditivity. Their criteria, based on a tail argumentation is developed below as follows:

- *Not too heavy tail*: if the risks have not too heavy tails, the authors see diversification as valid and VaR satisfies subadditivity in the tail;

- *Heavy tail*: if the risks have heavy tails, diversification may not be privileged and VaR may not be subadditive.

Finally, some authors point out that subadditivity (or its pendant for capital allocation, *no undercut*) may not be desirable, see Laeven and Goovaerts (2004) and Kim and Hardy (2009).

However, this argument that has been sometimes used to reject VaR for the benefit of TVaR has to be put into perspective, as researchers show that this is not that simple. One simple counter argument is that VaR is subadditive for elliptical distribution, as shown by Mc Neil, Frey and Embrechts (2005). However this is a limited argument as normality is not mainstream both in actuarial science (see e.g. Sauvet (2006), Heep-Altiner, Kaya, Krenzlin and Welter (2010) or Boland (2007) and in finance (see e.g. Mandelbrot and Hudson (2009), Walter and de Pracontal (2009) or Walter (2010)).

More appealing is the practical argument that in the tail region, typically at a level around $\alpha = 99\%$, VaR is roughly subadditive. This important rehabilitating argument for the use of VaR is developed by Denuit, Dhaene, Gooveaerts, Kaas and Laeven (2006) or Verlaine (2008) referring to the fact that VaR is subadditive for the relevant tail of heavy tail distributions, provided that these heavy tails are not *too super heavy*. Inui, Kijima and Kitano (2005) demonstrate that VaR has a considerable bias when used with heavy-tailed distributions. They show that the bias is increased at higher confidence level, heavier tails and smaller sample sizes. Danielsson, Samorodnitsky, Sarma, Jorgensen and Vries (2011) explore the potential violation of subaddivity by VaR, not only in theoretical terms but also through simulations. They conclude that VaR is deemed to be subadditive for a majority of applications, henceforth making unnecessary the research for an alternative risk measure solely because of the subadditivity argument.

Heyde et al. (2009) insist that VaR can be seen as a valid risk measure especially in the context of satisfying the regulatory needs, which is one of the angles of this work in the context of Solvency 2 regime. They however leave the choice to a company to use it for internal management purposes, whereby VaR might not seem to correspond to their preference.

4.3.3 TVaR and capital allocation

In order to derive the result of this subsection, two main sources may be identified in the literature. On the one hand Tasche (2000) may be used again like for the preceding subsection. Tasche identifies that he finds the same result as the one of Overbeck and Stahl (1998) albeit Tasche's form is slightly more general. On the other hand the axiomatic of Kalkbrener (2005) is another mean to arrive at the same derivative for TVaR, of which a presentation can also be found in Urban (2002). In the following Tasche's approach is presented only. Like for VaR, the quantile function may not be differentiable. Nonetheless Lemma 4.2 uses the same Assumption 4.1 stated before.

Lemma 4.2 (Partial derivative of) *Let* $(X_1,\ldots,X_n)$ *and* α *be as in Lemma 4.1 and assume*

$$E[|X_i|] < \infty, i = 1,\ldots,n.$$

Define U, Z_λ *and* $VaR_\alpha(Z_\lambda)$ *as in Lemma 4.1 and set* $\mathrm{TVaR}_\alpha(Z_\lambda)$ *so as to:*

$$TVaR_\alpha(Z_\lambda) = E[Z_\lambda \mid Z_\lambda \geq VaR_\alpha(Z_\lambda)] \text{ for } \lambda \in U.$$

Then $TVaR_\alpha(Z_\lambda)$ *is continuous and partially differentiable in* $\lambda_i, i = 1,\ldots,n$ *with continuous derivatives*

$$\frac{\partial TVaR_\alpha(Z_\lambda)}{\partial \lambda_i}(\lambda) = E[X_i \mid Z_\lambda \geq VaR_\alpha(Z_\lambda)] \text{ for } i = 1,\ldots,n. \tag{6}$$

Proof *See Tasche (2000).*

4.3.4 On the coherence of TVaR

As seen above, VaR is in most of the cases not subadditive, and thus not coherent in the axiomatic set of Arztner et al.. This is why some authors propose to work with measures like TVaR, because it fulfills the missing axiom of subadditivity unlike VaR and is therefore a coherent risk measure. Then TVaR is coherent, see e.g. Zhang and Rachev (2006), Acerbi and Tasche (2002), Scherpereel (2005) or Haugh (2010).

5. ECONOMIC CAPITAL AND SOLVENCY: IMPLEMENTING CAPITAL ALLOCATION

Aim of article 120 of the Solvency 2 Directive (2009) is that the internal model should not only enable to compute the regulatory economic capital, but also to be widely used by the company. The necessity to show how embedded an internal model is, has hence to be demonstrated by each company and is called the *use test* of an internal model. Capital allocation is one of them.

Moreover, as documented by Equation (2), economic capital for solvency calculation is derived from a risk measure. Under Solvency 2, the calculation are deemed to use VaR, however TVaR is not excluded when a company chooses to use an internal model, e.g. in Swiss Solvency Test (SST). In the coming section, practical allocation using the game theoretic settings given so far is going to be shown for VaR and TVaR.

5.1 A practical example of economic capital allocation in non-life insurance

5.1.1 Ruhm-Mango-Kreps algorithm adapted to quantile-based capital allocation

As Equations (5) and (6) show, computing the allocation for these tail-based measures are conditional expectations. Regarding VaR, the interpretation of Equation (5) can be seen as the average loss value of portfolio number i when cumulated losses of the overall portfolio reach the quantile at level $1-\alpha$. For TVaR, Equation (6) can be seen as the average loss value of portfolio number i when cumulated losses of the overall portfolio are over the quantile at level $1-\alpha$.

Clark (2005) identifies that the procedure given by Ruhm and Mango (2003), Mango (2003) and Kreps (2005) is well-adapted to compute conditional expectation, hence Equations (5) and (6). The aim of this contribution is to demonstrate the link between Ruhm-Mango-Kreps algorithm and the Aumann-Shapley formula. In other words, to show the formal equivalence of the RMK algorithm with the Aumann-Shapley allocation, as D'Arcy (2011) or Kaye (2005) alludes to without explicitly showing it.

A heuristic presentation of the algorithm may be:

1. simulation of the possible outcomes of results by LOB, where the aggregate result of all LOBs by each simulation number represents the result of the company. As implicitly outlined by Table 4, these sums are the results of aggregation techniques like correlation matrix and/or copulae;
2. sort out all the aggregates of the LOBs, from the best to the worst result;
3. if VaR has been selected as a risk measure, compute the $(1-\alpha)$-quantile among the aggregates of LOBs, and select the negative outcomes only for the LOBs where there is a negative outcome. If TVaR is the selected risk measure, select all the simulation outcomes beyond $(1-\alpha)$ for the aggregates of LOBs, and select the negative outcomes only for the LOBs where there is a negative outcome;
4. allocation of the capital by LOB: averaging the selected losses for the LOB in terms of aggregated losses of the sum of all LOBs, leading in effect to the capital allocation by LOB. This exercise is done for each LOB.

This is the heuristic presentation, as presented by Ruhm, Mango and Kreps. It is interesting to give the algorithm of Holden (2008) for VaR and Tsanakas (2013) for TVaR.

Algorithm 5.1 (RMK algorithm for VaR) *Consider an company with* n *LOBs. Let* X_i^j *be the* j *th simulated result for the* i *th LOB for* $j=1,\ldots,m$. *Let* X^j *be the* j *th simulated company-wide result and* $Z_\lambda = \sum_{i=1}^n \lambda_i X_i$. *For a confidence level* $\alpha, m_\alpha = (1-\alpha)m$ *is an integer such that* $-X^{m_\alpha}$ *is an estimator of* $VaR_\alpha(Z_\lambda)$. *Let* $d_\alpha \le m_\alpha$.

1. *Sort aggregate results to get* $X^{(1)},\ldots,X^{(m)}$.
2. *Estimate the VaR by* $-X^{m_\alpha}$.
3. *Compute*

$$m_\alpha = -\frac{1}{2d_\alpha+1} X^{(m_\alpha)} \sum_{j=-d_\alpha}^{d_\alpha} \frac{X_i^{(m_{\alpha+j})}}{X^{(m_{\alpha+j})}}$$

which estimates $E[X_i \mid Z_\lambda = VaR_\alpha(Z_\lambda)]$.

For a way to determine d_α, see e.g. Korn, Korn and Kroisandt (2010).

Algorithm 5.2 (RMK algorithm for TVaR) *Consider an company with* n *LOBs. Let* X_i^j *be the* j *th simulated result for the* i *th LOB for* $j=1,\ldots,m$. *Let* X^j *be the* j *th simulated company-wide result and* $Z_\lambda = \sum_{i=1}^n \lambda_i X_i$. *For a confidence level* $\alpha, m_\alpha = (1-\alpha)m$ *is an integer such that* $-X^{m_\alpha}$ *is an estimator of* $VaR_\alpha(Z_\lambda)$. *Let* $d_\alpha \le m_\alpha$.

1. *Sort aggregate results to get* $X^{(1)},\ldots,X^{(m)}$.
2. *Estimate the VaR by* $-X^{m_\alpha}$.
3. *Set* $X_{j:m}^{(\star)}$ *be the selection of* $X^{(1)},X^{(2)},\ldots,X^{(m)}$ *exceeding* $-X^{m_\alpha}$.
4. *Compute*

$$\frac{1}{m-m_\alpha m} \sum_{j=m_\alpha m+1}^{m} X_{j:m}^{(\star)}$$

which estimates $E[X_i \mid Z_\lambda \ge VaR_\alpha(Z_\lambda)]$.

5.1.2 Scope of the study: internal models as per ICA and Solvency 2

We are going to present capital allocation computations as per the algorithm presented above. They are based on the data of a anonymized company in non-life insurance. This company is based in the UK, and used to compute the economic capital through the Internal Capital Assessment (ICA) regulation until 2010; see Sandström (2005) for a presentation of the UK-solvency regime. The company has used its Solvency 2 internal model for computing is economic capital in 2011. These calculations are only relating to underwriting risk, at $1-\alpha = 99{,}5\%$. The quantile is estimated as a result of 10,000 simulations of the underwriting result. In order to estimate VaR, 30 simulation around the 99,5% mark are used, as recommended by Parry et al. (2009). Regarding TVaR estimation, exceeding 99,5% mark means using 50 simulations. The comparison between underlying modelling of underwriting risk between the ICA and Solvency 2 internal models for the sample company are provided in the Appendix under Table 4.

The following computations have been made:

Case 1: Calculation under the ICA using a 5-year horizon;

Case 2: Calculation under the ICA using a 1-year horizon;

Case 3: Calculation under the Solvency 2 using a 1-year horizon;

Case 4: Calculation under the ICA using a 5-year horizon (rescaled);

Case 5: Calculation under the ICA using a 1-year horizon (rescaled);

Case 6: Calculation under the Solvency 2 using a 1-year horizon (rescaled).

The rescaling is achieved by removing the mean result from the calculations. In practical terms the mean of each year for all simulations is computed, then subtracted from the minimum result. The idea is to rescale from the short term volatility observed in the result. In a longer term view, the idea is to concentrate on a mean result, rescaled from the short term volatility.

For the ICA internal model, simulations have been provided by Simulum (Microsoft Excel add-on provided by the company Watson Wyatt), whereas for the Solvency 2 internal model, simulations come from Risk Explorer (edited by the company Ultimate Risk Solutions).

5.1.3 Results

A breakdown of detailed result are provided in Table 5 in the Appendix. They are the net technical results in £ million. Figures 1 and 2 below compare graphically the results per case, as per described for the eight LOBs of the company.

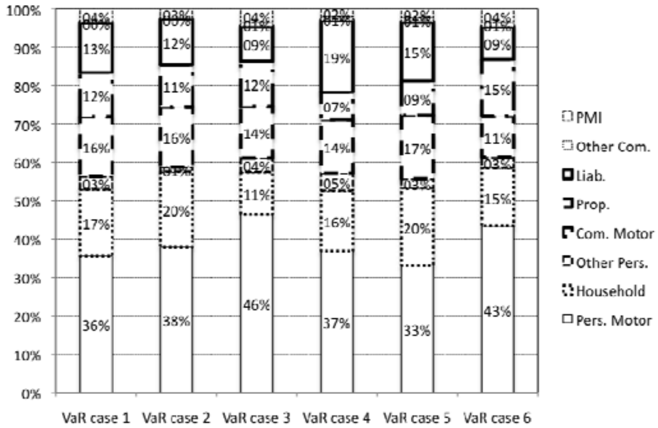

Figure 1: Comparison of allocation coefficients depending on underlying internal modelling assumption for VaR

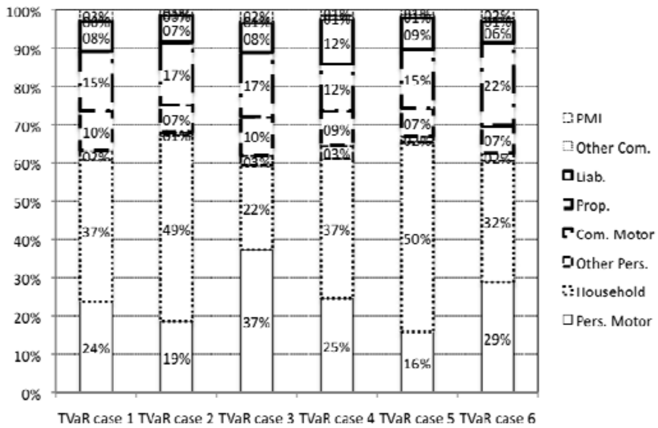

Figure 2: Comparison of allocation coefficients depending on underlying internal modelling assumption for TVaR

Next two Figures 3 and 4 plot the same results as the two preceding graphs, but aggregated by *macro-LOBs*. The macro-LOBs are as follows: Personal lines are the sum of Personal Motor, Household and Other Personal Lines, Commercial Lines are the sums of Commercial Lines, Property, Liability and Other Commercial.

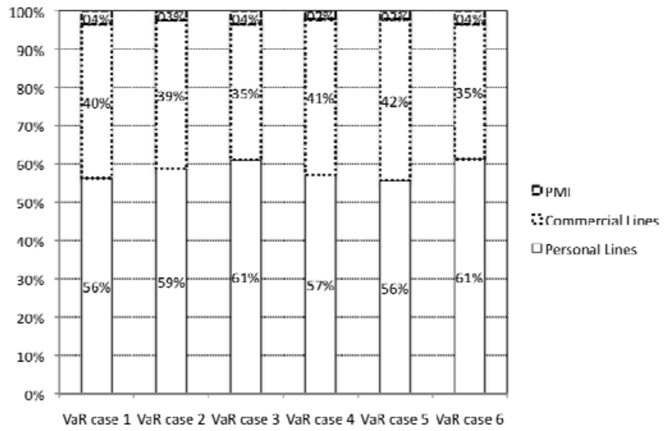

Figure 3: Comparison of allocation coefficients depending on underlying internal modelling assumption for VaR (macro-LOBs)

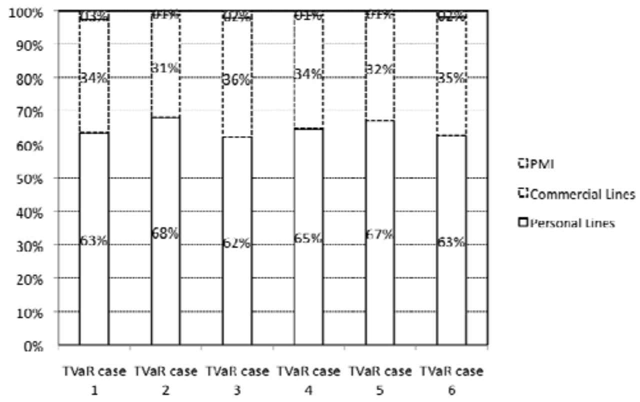

Figure 4: Comparison of allocation coefficients depending on underlying internal modelling assumption for TVaR (macro-LOBs)

Interpretation of these graphs are given in the next section.

5.2 Coherent allocation with a practical insight

Figures 1 and 2 highlight the fact that Household, especially under TVaR can get a very high allocation under the ICA modelling, reflecting the high volatility of the business. This is linked to fact that Household is catastrophe prone. The dichotomy between *catastrophe prone* and *non-catastrophe prone* in allocation context is also recognised by Vaughn (2007). Yet this statement does not totally hold under the Solvency 2 modelling (cases 3 and 6), whereby each LOB gets an allocation closer to its share in the business in terms of gross written premium. These shares are disclosed in Figure 5 of the Appendix. This shows that a careful attention has to be paid to modelling changes when it comes to capital allocation, as ICA modelling is not in this case exactly the same as the Solvency 2 calculation. Stölting (2003) recognises in particular for TVaR-based allocation that although it takes into account non-linear dependency and respects capital allocation coherency axioms, it may be sensitive to even small changes of modelling. In this respect Schmock and Strautmann (1999) remind how delicate it may be to manipulate TVaR for *small* portfolios and discrete distributions. Nonetheless, looking at the allocation between Personal and Commercial lines (Figures 3 and 4), there is no real transfer at the aggregated LOBs level, rather a reallocation within the aggregated LOBs. This is an important consideration at a management level, as even if it is assumed that Household and Personal Motor Directors ought to have in mind these patterns when underwriting, Personal lines Director may still enjoy the benefit of the diversification between the two LOBs.

A second comment is on practical considerations. There may be some struggle to internally communicate why Household gets such a high capital charge in one modelling, and less in a reviewed modelling. The general issue of communication in relation to capital allocation is stressed by Roberts (2006), chief actuary of RSA at the time of his presentation. Decupère (2011) highlights also this point. This may indicate a shift in the risk profile of the company and hence needs to be further investigated. So more generally it is always necessary to perform sanity checks on the results of the capital allocation and to link it to the risk profile of the company when explaining capital allocation results.

Another comment in the light of Figures 1 and 3 that the allocation coefficients are *relatively* stable regardless of the underlying internal model for VaR. This phenomenon is not reproduced for TVaR. In other words the coefficients seem more stable with VaR than with TVaR. This is an appealing argument in favour of VaR, which leads to more consistent results over time and whatever the model may be. This consistency argument is not to be neglected again in the perspective of internal communication with the management. Furthermore, it ties up neatly with other consistency arguments, as VaR as risk measure may be also used:

- for the total company's economic capital calculation. Holden (2008) recommends to use for the allocation the same risk measure as the one used for the economic capital calculation;

- for the modelling underlying the catastrophe reinsurance purchase decision;
- for the risk appetite of the sample company.

However it has to be kept in mind, that VaR results are based on a 1 in 200 years event only (corresponding to 99,5%). TVaR takes the mean of all large losses above this 99,5% point. We would therefore not dismiss TVaR, as it is possible to find many arguments and counter-arguments either for VaR or TVaR, see e.g. Sandström (2011) or Zec (2012). Moreover, SCOR use internal model to compute the Solvency Capital Requirement (SCR), whereas TVaR for the allocation, cf. Busse (2013). We propose additional arguments on VaR versus TVaR from a capital allocation perspective, synthetised in Table 3. The tick between brackets recalls that in some cases the property may be met. Obviously, these conclusions very much depend on the risk profile and historical technical results of the underlying portfolio used for the computation.

	VaR	TVaR
Coherent measure of risk	✗(✓)	✓
Coherent allocation of economic (risk) capital	✗(✓)	✓
Consistent allocation of economic (risk) capital	✓	✗

Table 3: Coherencies and consistencies of VaR and TVaR through risk measure and allocation spectra

There is hence an interest to have a mixed approach in terms of risk measure, even if a preference may emerge for the use of VaR. A last comment binding with the preceding one is that we compare VaR and TVaR for the level $1-\alpha$. It is known for a normal distribution that the two measures are comparable for different level of $1-\alpha$. Comité Européen des Assurances (CEA) reports that a 99,5% VaR is equivalent to a 98,7% TVaR, and that a 99% TVaR is equivalent to a 99,62% VaR, cf. CEA (2006). The difference between the $1-\alpha$ of VaR and TVaR would be higher for more skewed distributions than the normal one. As we are not dealing with a normal distribution but with a mix of distributions in the internal model, a capital allocation model development could be to find where the comparable point is to be set, if we seek to enhance the TVaR aspect of it.

6. CONCLUSION

The purpose of this paper is to give new theoretical and practical insights on capital allocation. We highlight the relationship between a coherent risk measure and a coherent capital allocation by using cooperative game theory and concept of risk measures. This is enabled by the Aumann-Shapley value, as highlighted by Denault, unlike the Shapley value. Although Denault sketched the relationship with coherent measure of risk, Buch and Dorfleitner have made this relationship clear. Then, when it comes to application, it is possible to use tail-based measures, commonly used for computing economic capital. Such an application can be done through simulation techniques, and in this perspective RMK algorithm can be used. The study laid out here show that although VaR is not commonly known as a coherent measure of risk, and such a statement may be discussed, TVaR may not be the perfect alternative. This may be down to the risk profile of the portfolio used, but it comes out that from an allocation viewpoint, VaR is more *consistent* through the time. This is indeed important when one has to think about change in internal modelling and the effect on capital allocation, which has been investigated in this paper.

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7. APPENDIX

7.1 Underlying modelling for the ICA and Solvency 2 internal models of the sample company

Please see Table 4 hereunder.

Part	ICA[a]	Solvency 2
Modelled classes	Corresponds to the company's granularity (e.g. Private Car, Motorcycle, Commercial Vehicle)	Can correspond to Company's granularity. However, regulation can be more prescriptive, e.g. Motor classes have to be split between Third Parties' Liability and Own Damages
Attritional claims	Frequency/severity approach. Claim frequency[b]: normal distribution, claim size: gamma distribution	Aggregate loss ratio distribution (lognormal)

Large claims	Frequency/severity approach (collective model). Claim frequency: Poisson distribution, claim severity: Pareto distribution	Frequency/severity approach (collective model). Claims frequency: negative binomial distribution and claim severity: generalised Pareto distribution
Reserve variability	Mack model associated with a loss distribution assumption	Merz and Wüthrich model associated with a loss distribution assumption
Discounting present value[c]	May be applied according to the regulation, however there is an option not to do so, explicitly allowing for an investment result	Must be applied, i.e. all cash flows are present valued, using risk free rate curve (market value basis)
Other insurance risk	Catastrophe risk, inflation shock, bodily injury surge	Catastrophe risk (more granular than the ICA model), inflation shock, bodily injury surge
Correlations	Internal design (correlation matrices between LOBs)	Follows a more prescriptive approach according to the regulation (mix of correlation matrices -from standard formula- and copulae), with different levels of aggregation
Horizon	Business written in year 1, and then run off ; computation up to a 5 year time horizon	Business earned in year 1 only; computation over 1 year time horizon

[a] Example for Personal Motor.

[b] Expressed as a percentage and not as a number.

[c] The general principle is that insurance risk can be reduced by investment income that will be earned on assets held (cf. e.g. Lloyd's in Parry et al. (2009)). Under Solvency 2, the best estimate technical provisions are discounted, before adding up risk margin on it.

Table 4: Comparison of ICAS and Solvency 2 internal models

7.2 Allocation percentages by LOB for each underlying modelling

Please see Table 5 hereunder. Subscripts of VaR and TVaR designate the case, i.e. VaR_1 means "allocation coefficients for case 1 for VaR".

	Personal lines			Commercial lines				PMI	Total
	Motor	House- hold	Other Pers.	Motor	Prop.	Liab.	Other Com.		
VaR_1	35,6%	17,3%	3,4%	15,5%	11,6%	13,0%	0,1%	3,5%	100%
	56,3%			40,2%					
$TVaR_1$	23,8%	37,1%	2,4%	10,4%	15,4%	8,1%	0,1%	2,6%	100%
	63,3%			34,0%					
VaR_2	37,8%	19,7%	1,2%	15,8%	11,0%	11,8%	0,2%	2,5%	100%
	58,7%			38,8%					
$TVaR_2$	18,7%	48,7%	0,8%	7,1%	16,5%	7,0%	0,0%	1,3%	100%
	68,1%			30,6%					
VaR_3	46,4%	11,0%	3,7%	13,5%	11,7%	9,2%	1%	3,5%	100%
	61,1%			35,4%					
$TVaR_3$	37,4%	22,0%	2,7%	10.2%	16,7%	7,9%	0,8%	2,3%	100%
	62.1%			35.6%					
VaR_4	36,8%	15,7%	4,6%	14,0%	7,2%	18,7%	0,9%	2,1%	100%
	57,1%			40,8%					
$TVaR_4$	24,7%	36,6%	3,4%	9,1%	12,2%	11,8%	0,9%	1,4%	100%
	64,6%			34,0%					
VaR_5	33,0%	20,2%	2,5%	16,5%	9,0%	15,4%	1,1%	2,3%	100%
	55,7%			42%					
$TVaR_5$	15,8%	49,6%	1,7%	7,3%	15,3%	8,6%	0,5%	1,2%	100%
	67,1%			31,7%					
VaR_6	43,3%	15,2%	2,7%	10,8%	14,7%	8,6%	1,1%	3,5%	100%
	61,3%			35,2%					
$TVaR_6$	28,9%	31,7%	2,0%	7,2%	21,8%	5,7%	0,7%	2%	100%
	62,6%			35,4%					

Table 5: Breakdown of allocation coefficients for the six cases for VaR and TVaR (per case, first row for each LOB, second row for macro-LOBs)

7.3 LOB part as a share of gross written premiums

The pie chart of Table 5 shows the share of each LOB in terms of gross written premium for the company (average of the history 2006-2011).

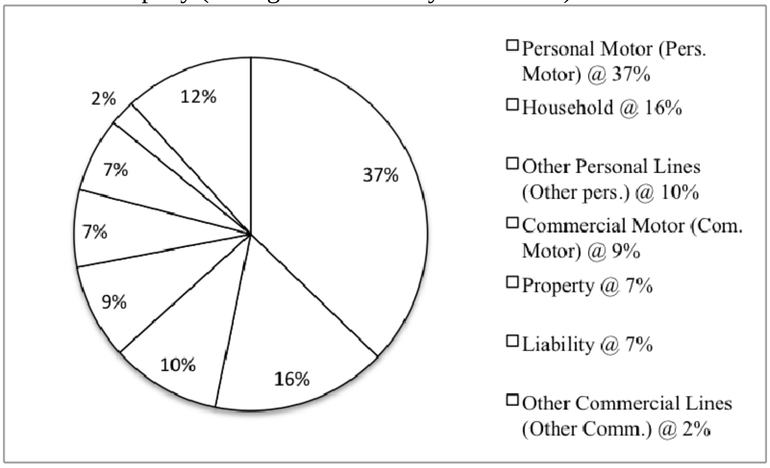

Figure 5: LOB part as a share of gross written premiums

REFERENCES

ACERBI, C., & TASCHE, D. (2002). On the coherence of Expected Shortfall. *Journal of Banking & Finance*, *26* (7), 1487-1503.

ALBRECHT, P. (2003, February). *Risk based capital allocation* (Tech. Rep. No. 03-02). Sonderforschungsbereich 504, Universität Mannheim.

ALBRECHT, P., & KORYCIORZ, S. (2004). Methoden der risikobasierten Kapitalallokation im Versicherungs- und Finanzwesen. *Zeitschrift für die gesamte Versicherungswissenschaft*, *93* (2), 123-159.

ARTZNER, P. (1999, April). Application of Coherent Risk Measures to Capital Requirements in Insurance. *North American Actuarial Journal*, *3* (2), 11- 25.

ARTZNER, P., DELBAEN, F., EBER, J.-M., & HEATH, D. (1999, July). Coherent Measures of Risk. *Mathematical Finance*, *9* (3), 203-228.

AUBIN, J.-P. (1979). *Mathematical methods of game and economic theory*. North-Holland Publishing Co.

AUBIN, J.-P. (1981). Cooperative fuzzy games. *Mathematics of Operations Research*, *6* (1), 1-13.

AUMANN, R., & SHAPLEY, L. S. (1974). *Value of non-atomic games*. Princeton, New Jersey: Princeton University Press.

BALOG, D. (2010). Risk based capital allocation. In L. Á. Kóczy (Ed.), *Proceedings of fikusz 2010* (p. 17-26). Budapest: Keleti Károly Faculty of Business and Management, Óbuda University.

BILLERA, L. J., & HEATH, D. C. (1982). Allocation of shared costs: a set of axioms yielding a unique procedure. *Mathematics of Operations Research*, *7* (1), 32-39.

BILLERA, L. J., & RAANAN, J. (1981). Cores of nonatomic linear production games. *Mathematics of Operations Research*, *6* (3), 420-423.

BOLAND, P. J. (2007). *Statistical and probabilistic methods in actuarial science*. Boca Raton, Florida: Chapman & Hall/CRC.

BORCH, K. (1962). Application of game theory to some problems in automobile insurance. *ASTIN Bulletin, 2* , 209-221.

BUCH, A., & DORFLEITNER, G. (2008). Coherent risk measures, coherent capital allocations and the gradient allocation principle. *Insurance Mathematics and Economics*, *42* (1), 235-242.
Available from http://linkinghub.elsevier.com/retrieve/pii/S0167668707000200

BUSSE, M. (6 June 2013). SCOR's Internal Model and its application to Solvency II. In *10th congress of national actuaries.* Rome.

CADOUX, D., LOIZEAU, J.-M., & PARTRAT, C. (2003, 5 and 6 November). Mesures de risque, besoin en fonds propres. In *Séminaire solvabilité et rentabilité en assurance non vie.*

CEA. (7 November 2006). *CEA Working Paper on the risk measures VaR and TailVaR* (Tech. Rep.). Brussels: Comité Européen des Assurances.

CHARPENTIER, A. (2007, 12 June). Utilisation des copules et conséquences sur le besoin en capital. In *Mesures de risque, dépendance stochastique et allocation de capital.*

CLARK, D. R. (2005, Spring). Reinsurance Applications for the RMK Framework. In *Casualty actuarial society forum* (p. 353-366). Casualty Actuarial Society.

COURNOT, A. (1838). *Recherches sur les Principes Mathématiques de la Théorie des Richesses*. Paris: Hachette.

CUMMINS, J. D., LIN, Y., & PHILLIPS, R. D. (21 June 2006). Capital Allocation and the Pricing of Financially Intermediated Risks: An Empirical Investigation. *working paper*.

DANIELSSON, J., SAMORODNITSKY, G., SARMA, M., JORGENSEN, B. N., & VRIES, C. G. de. (2011, April). Fat Tails, VaR and Subadditivity. Working paper.

D'ARCY, S. P. (2011, 16 March). Capital Allocation in the Property-Liability Insurance Industry. In *2011 ERM Symposium* (Vols. "Strategic Risk, Value Creation, Risk Indices, and the Cost of Capital Sustainability of Earnings" session). Chicago.

DAUGHETY, A. F. (Ed.). (1988). *Cournot oligopoly - Characterization and applications*. Cambridge University Press.

DE WEERT, F. (2011). *Bank and insurance capital management*. Wiley.

DECUPERE, S. (2011). *Agrégation des risques et allocation de capital sous Solvabilité II*. Unpublished master's thesis, Ecole Nationale de la Statistique et de l'Administration Economique, Malakoff.

DELBAEN, F. (2002). Coherent Risk Measures on General Probability Spaces. In K. Sandmann & P. J. Schönbucher (Eds.), *Advances in finance and stochastics: essays in honour of Dieter Sondermann* (p. 1-37). Berlin:Springer.

DENAULT, M. (2001). Coherent allocation of risk capital. *Journal of Risk*, *4* (1), 1-34.

DENUIT, M., DHAENE, J., GOOVAERTS, M. J., & KAAS, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. Chichester: John Wiley & Sons.

DENUIT, M., DHAENE, J., GOOVAERTS, M. J., KAAS, R., & LAEVEN, R. (2006). Risk measurement with equivalent utility principles. *In: Rüschendorf, Ludger (Ed.), Risk Measures: General Aspects and Applications (special issue), Statistics and Decisions*, 1-26.

DERIEN, A., & LE FLOC'H, E. (2011, December). L'analyse de la rentabilité vue par la formule standard. *Bulletin Français d'Actuariat*, *11* (22), 83-104.

DIERS, D. (2007). Die Wirkung unterschiedlicher Risikokapitalallokationsmethoden - Übersicht und Fallstudie. In *Preprint series of the faculty of mathematics and economics* (Vol. 4). Universität Ulm.

Directive. (2009). *Directive of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II)(recast)*. Directive 2009/138/EC.

DOFF, R. (2007). *Risk Management for Insurers - Risk Control, Economic Capital and Solvency II*. Risk Books.

ELING, M., & SCHMEISER, H. (2010, 8 November). *The Risk Premium Project (RPP) Update - RPP II Report* (Tech. Rep.). Casualty Actuarial Society–Committee on Theory of Risk.

FISCHER, T. (2003). Risk capital allocation by coherent risk measures based on one-sided moments. *Insurance: Mathematics and Economics, 32* (1), 135-146.

FISCHER, T. (2004). *Valuation and risk management in life insurance*. Unpublished doctoral dissertation, Fachbereich Mathematik der Technischen Universität Darmstadt.

GOURIEROUX, C., LAURENT, J.-P., & SCAILLET, O. (2000). Sensitivity Analysis of Values at Risk. *Journal of Empirical Finance*, *7* , 225-245.

GOURIÉROUX, C., & MONFORT, A. (2011, 29-30 September). Allocating Systematic and Unsystematic Risks in a Regulatory Perspective. In *Stability and Risk Control in Banking, Insurance and Financial Markets, 10th International Conference on Credit Risk Evaluation.* Venice.

GRÜNDL, H., & SCHMEISER, H. (2007, January). "Capital Allocation in Insurance: Economic Capital and the Allocation of the Default Option Value" by Michael Sherris and John van der Hoek, April 2006. In *Discussions of Papers Already Published with Authors' Reply* (Vol. 11, p. 163-165). North American Actuarial Journal.

HALLERBACH, W. G. (1999). Decomposing Portfolio Value-at-Risk: a General Analysis. *Tinbergen Institute Rotterdam*, *29* (99-034/2).

HAMLEN, S. S., HAMLEN, W. A., & TSCHIRHART, J. T. (1977). The Use of Core Theory in Evaluating Joint Cost Allocation Schemes. *The Accounting Review*, *52* (3), 616-627.

HAUGH, M. (2010, Spring). *Risk Measures, Risk Aggregation and Capital Allocation - Quantitative Risk Management.*

HEEP-ALTINER, M., KAYA, H., KRENZLIN, B., & WELTER, D. (2010). *Interne Modelle nach Solvency II - Schritt für Schritt zum internen Modell in der Schadenversicherung*. Karlsruhe: Verlag Versicherungswirtschaft.

HEYDE, C. C., KOU, S., & PENG, X. (2009, February). *What Is a Good External Risk Measure: Bridging the Gaps between Robustness, Subadditivity, and Insurance Risk Measures.* (Working Paper)

HOLDEN, L. (2008). *Some properties of Euler Capital Allocation.* (Submitted to The Journal of Risk)

INUI, K., KIJIMA, M., & KITANO, A. (2005). VaR is subject to a significant positive bias. *Statistics & Probability Letters*, *72* (4), 299-311.

JORION, P. (2007). *Value at Risk: The New Benchmark for Managing Financial Risk* (Third ed.). McGraw-Hill.

KALKBRENER, M. (2005). An axiomatic approach to capital allocation. *Mathematical Finance*, *15* (3), 425-437.

KANEKO, M., & WOODERS, M. H. (2004). Utility theories in cooperative games. In (Kluwer Academic Press ed., Vol. 2, p. 1065-1098). Salvador Barberà, Peter Hammond and Christian Seidl.

KAYE, P. (2005). *Risk Measurement in Insurance - A Guide To Risk Measurement, Capital Allocation And Related Decision Support Issues* (Discussion Paper Program). Casualty Actuarial Society.
Available from http://www.casact.org/pubs/dpp/dpp05/05dpp1.pdf

KIM, J. H.-T. (2007). *Estimation and allocation of insurance risk capital*. Unpublished doctoral dissertation, University of Waterloo, Waterloo, Ontario.

KIM, J. H.-T., & Hardy, M. R. (2009). A capital allocation based on a solvency exchange option. *Insurance: Mathematics & Economics*, *44* (3), 357-366.

KORN, R., KORN, E., & KROISANDT, G. (2010). *Monte carlo methods and models in finance and insurance*. Chapman & Hall/CRC.

KORYCIORZ, S. (2004). *Sicherheitskapitalbestimmung und -allokation in der Schadenversicherung - Eine risikotheoretische Analyse auf der Basis des Value-at-Risk und des Conditional Value-at-Risk* (V. Versicherungswirtschaft, Ed.). Karlsruhe: Peter Albrecht and Egon Lorenz.

KREPS, R. E. (2005). Riskiness Leverage Models. In *Proceedings of the casualty actuarial society* (Vol. XCII, p. 31-60). Arlington, Virginia.

LAEVEN, R. J. A., & GOOVAERTS, M. J. (2004). An optimization approach to the dynamic allocation of economic capital. *Insurance: Mathematics & Economics, 35* (2), 299-319.

LEMAIRE, J. (1985). *Automobile insurance: Actuarial models*. Boston: Kluwer.

LEMAIRE, J. (1991). Cooperative game theory and its insurance applications. *ASTIN Bulletin*, *21* (1), 17-40.

LEYTON-BROWN, K. (2008a). *Coalitional game theory.* Vancouver.

LEYTON-BROWN, K. (2008b). *Self-interested agents and utility theory.* Vancouver.

LONG, P., & WHITWORTH, J. (2004). Risk diversification benefits in Insurance - How to recognise them and what to do with them. In *Life convention 2004.*

MANDELBROT, B., & HUDSON, R. L. (February 2009). *Une approche fractale des marchés - Risquer, perdre et gagner* (Marcel Filoche translated ed.). Odile Jacob.

MANDL, J. (2005). Spieltheoretische Verfahren der Kapitalallokation im Versicherungsunternehmen. *Zeitschrift für die gesamte Versicherungswissenschaft, 94* , 79-103.

MANGO, D. F. (2003). Capital Consumption: An Alternative Methodology for Pricing Reinsurance. In *ASTIN Colloquium International Actuarial Association.*

MCCROSSAN, J., Manley, C., & Lavelle, D. (2006, 27 April). RBC and its Practical Applications to Non-Life Insurance. In *Evening meeting.* Alexander Hotel, Dublin.

MCNEIL, A., Frey, R., & Embrechts, P. (2005). *Quantitative risk management :concepts, techniques, and tools*. Princeton University Press.

MILDENHALL, S. J. (2004). A note on the Myers and Read capital allocation formula. *North American Actuarial Journal*, *8* (2), 32-44.

MILDENHALL, S. J. (2006, 21-23 April). Actuarial Geometry. In *Risk Theory Society Annual Seminar*. Richmond, Virginia.

MIRMAN, L. J., & TAUMAN, Y. (1982). Demand compatible equitable cost sharing prices. *Mathematics of Operations Research*, *7* (1), 40-56.

OSBORNE, M. J., & RUBINSTEIN, A. (1994). *A course in game theory*. The MIT Press.

OVERBECK, L., & STAHL, G. (1998). Stochastische Modelle im Risikomanagement des Kreditportfolios. In A. Oehler (Ed.), *Credit Risk und Value-at-Risk Alternativen* (p. 77-110). Stuttgart: Schäffer-Poeschel Verlag.

PARRY, J., Brown, E., Indge, D., Johnson, H., Reeves, O., & Kirk, J. (2009, 24 March). *ICA - 2010 minimum standards and guidance* (Tech. Rep.). Lloyd's.

POLLACK, S. (2006, April). Reinsurance is an easy game. *The Actuary*, 24-26.

POWERS, M. R. (2007, July). Using Aumann-Shapley Values to Allocate Insurance Risk: The Case of Inhomogeneous Losses. *North American Actuarial Journal*, *11* (3), 113-127.

RAFELS, C. (2006). *Monotonic and core selections in cooperative games with transferable utility* (Tech. Rep.). University of Barcelona.

ROBERTS, D. (2006, 26 September). Capital Allocation in the Insurance Company. In *Asset and liability management association meeting.*

ROOTZÉN, H., & KLÜPPELBERG, C. (1999). A single number cannot hedge against economic catastrophes. In *Ambio. Special Issue on Risk Assessment - An International and National Perspective* (p. 550-555). Royal Swedish Academy of Science.

RUHM, D., & MANGO, D. F. (2003). A Risk Charge Calculation Based on Conditional Probability. In *2003 Thomas P. Bowles Jr. Symposium.* Casualty Actuarial Society.

SANDSTRÖM, A. (2005). *Solvency: Models, Assessment and Regulation*. Chapman & Hall/CRC.

SANDSTRÖM, A. (2011). *Handbook of solvency for actuaries and risk managers: Theory and practice*. Chapman & Hall/CRC.

SAUVET, C. (21 February 2006). *Solvency II - Quelle modélisation stochastique des provisions techniques prévoyance et non vie ?* Unpublished master's thesis, Institut de Science Financière et d'Assurances, Université Claude Bernard Lyon I.

SCHERPEREEL, P. (2005). *Risikokapitalallokation in dezentral organisierten Unternehmen* (C. Homburg, Ed.). DUV Gabler Edition Wissenschaft.

SCHLEE, W. (1999, Second half term). *Vorlesung über* Spiel*theorie.* (Technische Universität München)

SCHMOCK, U., & STRAUMANN, D. (1999, 15, 17 and 19 July). Allocation of risk capital and performance measurement. In *Conference on quantitative methods in finance.* Sydney.

SCHRADIN, H. R., & ZONS, M. (2003). Determination and allocation of riskadequate equity capitalization for performance measurement. In *Mitteilung 2/2003 des Instituts für Versicherungswissenschaft an der Universität zu Köln.* Cologne.

SHAPLEY, L. S. (1953). A value for *n*-person games. In H. Kuhn & A. Tucker (Eds.), *Contributions to the theory of games* (Vol. II, p. 307-317). Princeton University Press.

SHAPLEY, L. S. (1971-1972). Cores of Convex Games. *International Journal of Game Theory, 1* , 11-26.

SIMON, C. P., & BLUME, L. (2008). *Mathématiques pour économistes*. Bruxelles: de Boeck.

STÖLTING, R. (2003, 2 June). Risikokapital - Definition und Allokation. In *Versicherungsmathematischen Kolloquiums der Universität zu Köln.*

TASCHE, D. (2000, 18 February). *Risk contributions and performance measurement.* (Working paper of the Zentrum Mathematik, Technische Universität München)

TOUFFUT, J.-P. (Ed.). (2007). *Augustin Cournot: Modelling Economics*. Paris: Cournot Center for Economic Studies.

TSANAKAS, A. (2009). To split or not to split: Capital allocation with convex risk measures. *Insurance: Mathematics & Economics*, *44* (2), 268-277.

TSANAKAS, A. (2013). *Marginal allocations for VaR and TVaR, Enterprise Risk Modelling Notes, support pages for courses taught at Cass Business School.* Retrieved 1

November 2013, from http://casserm.wordpress.com/lecture-notes/5-measuring-risk-and-performance/512-marginal-allocations-for-var-and-tvar/

URBAN, M. (2002). *Allokation von Risikokapital auf Versicherungsportfolios*. Unpublished master's thesis, Zentrum Mathematik, Technische Universität München.

VAUGHN, T. R. (2007, Winter). Comparison of Risk Allocation Methods - Bohra-Weist

DFAIC Distributions. In *Casualty actuarial society forum* (p. 329-337). Casualty Actuarial Society.

VENTER, G. G. (19 January 2002). Risk Adjusted Profitability by Business Unit: How to Allocate Capital and How Not to. http://www.cap.columbia.edu/Venter.pps.

VENTER, G. G. (2004). Capital Allocation Survey with Commentary. *North American Actuarial Journal*, *8* (2), 96-107.

VENTER, G. G. (March 2007). Strategic Planning Models. *Risk Management Newsletter*, *10* (1), 22-25.

VERLAINE, M. (2008, November). Problématiques d'allocation du capital économique sous Bâle II . *AGEFI Luxembourg*.

WALTER, C. (2010). Le phénomène leptokurtique. In *Nouvelles normes financières. s'organiser face à la crise* (p. 59-79). Paris: Directed by Christian Walter. Springer.

WALTER, C., & de Pracontal, M. (October 2009). *Le Virus B - Crise financière et mathématiques*. Editions du Seuil.

ZADEH, L. A. (1965). Fuzzy sets. *Information and Control*, *8* , 338-353.

ZEC, N. (2012). *Use of an internal model in a general insurance company: Focus on economic capital allocation*. Unpublished master's thesis, Centre d'Etudes Actuarielles, http://bit.ly/11ODFDW.

ZHANG, Y. (2008). Allocation of capital beween assets and liabilities. *ASTIN Bulletin*, *38* (1), 1-11.

ZHANG, Y., & RACHEV, S. (2006). Risk Attribution and Portfolio Performance Measurement: An Overview. *Journal of Applied Functional Analysis*, *1* (4), 373-402.