

PROFIT SHARING: A STOCHASTIC CONTROL APPROACH

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ABSTRACT:

A majority of life insurance contracts encompasses a guaranteed interest rate and a participation to earnings of the insurance company. This participation, called profit share, is usually commuted into an increase of benefits. Furthermore, the amount distributed as a profit share is freely chosen by insurers in most of European countries. The insurer's decision to grant a profit share or not is in this context, influenced by the competition on the market and by shareholders' waitings. This paper proposes a method adapted to this situation, to optimize both the profit shares distribution and the asset allocation, based upon a stochastic control approach. In this setting, optimal strategies are those maximizing the expected economic utilities of future profit shares and of the insurer's future economic wealth.

Keywords : life insurance, profit sharing, stochastic control.

1. INTRODUCTION

A wide majority of life insurance policies are sold with an annual guaranteed return. However, this guaranteed rate being relatively lower than the market performance (and upper bounded by regulators), insurance companies redistribute a part of their earnings to insureds. This profit share generally increases the amount of benefits and is capitalized at the initial guaranteed rate. The level of profit shares is freely chosen by insurers in many countries (e.g. Belgium, Netherlands,...) and in fact depends mainly on the market competition and on the shareholders' waitings.

The starting point of many recent studies on life participating policies is the model of Briys & de Varenne (1997a, 1997b), that aims to value the market prices of the guarantee and of the profit sharing system. The guarantee and profit share are assimilated to options on future insurer's earnings and are priced with the Black & Scholes formula. The optimal profit share is set such that commercial loadings cover those option prices. Miltersen and Persson (2003) have developed a multi periods extension of this model. Whereas Bacinello (2001) has valued the profit sharing option, but taking explicitly into account the mortality risk.

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Grosen et Jørgensen (2000) have valued the cost of profit shares by a Monte Carlo approach. Jørgensen (2001), Grosen and Jørgensen (2002) have furthermore shown that a participating policy may be seen as the sum of four components: a zero coupon bond, a bonus option, a put option linked to the insurer's risk of default and an anticipative endowment in case of default before term. In papers of Bernard et al. (2005), as in Grosen & Jørgensen (2002), the possibility of an anticipative payment is considered. One also refers to Jensen et al. (2001) who have used a finite difference approach to price the profit sharing option.

Rather than trying to determine the optimal profit share by the option theory, this paper explores an alternative way based upon stochastic control. In this setting, one seeks the profit sharing and investment strategies that maximize economic utilities of future profit shares and of future insurer's wealth. The proposed model also allows to study the allocation of profit shares between contracts having different interest rate guarantees and maturities.

The outline of the paper is as follows. Section 2 presents an accounting method of contract in term of units of account. This will allow us to interpret the profit share as an increase of units rather than an increase of guaranteed benefits. Next, the dynamic of assets is detailed. In section 3, the optimization problem is set up. The Hamilton Bellman Jacobi (HJB) equation related to our model is detailed and solved in section 4. Finally, we end up this paper by a numerical example.

2. LIABILITIES AND ASSETS.

In this paper, we consider the case of an insurer holding a portfolio of L participating policies, on the liability side of his balance sheet. To avoid complication, the participating policies are here simple capitalization products with an unique premium paid in at the issuance. The profit shares distributed along the lifetime of a policy increase the capital delivered at maturity. The guaranteed interested rate and the maturity coupled to the policy number $i \in \{1 \dots L\}$ are respectively noted r_i^g and T_i . The benefits of the i^{th} contract are accounted in term of units of account, as for unit linked products. The i^{th} policyholders owns at time t , n_t^i units of account and one unit of account delivers a capital C_i at maturity.

The mathematical provision¹ of one unit of the i^{th} contract is therefore calculated as:

¹The mathematical provision is the debt of the insurer toward the policyholder.

$$R_t^i = e^{-r_i^g(T_i-t)} C_i \quad i = 1 \dots L$$

The total mathematical provision hold by the i^{th} insured is simply equal to the number of units times the provisions of one unit: $n_t^i R_t^i$. As illustrated in figure 1, this amount is accounted on the liability side and represents the commitment of the insurer to deliver a capital $n_t^i C_i$ at maturity.

The profit shares are assumed to be paid continuously. P_t^i denotes the participation paid at time t to the i^{th} policy. Those are the first parameters of control of our model. The formulation in continuous time makes the reading of results easier than in discrete time. However, the results presented in next sections may be discretized without loss of generality. The accounting in units of accounts allows us to commute the profit share P_t^i as an increase of units n_t^i . As illustrated in figure 0, the relation linking the variation of units of account to the participation is the following:

$$dn_t^i = \frac{P_t^i}{R_t^i} dt \quad i = 1 \dots L \tag{1}$$

Note that our approach to model liabilities may be extended to policies with multiple payments.

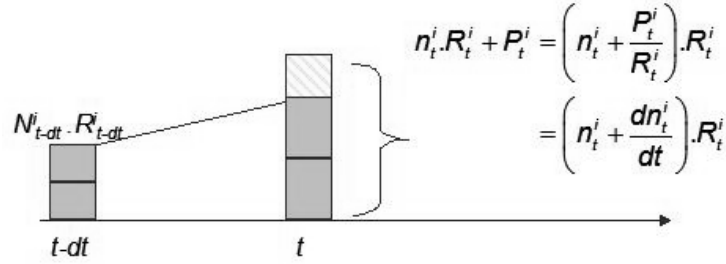


Figure 1: Commutation of Profit shares into units of account.

We assume that the insurer invests his holdings in a risk free rate asset and in a risky asset, like a stock. The risk free rate is constant and noted r . The risky asset, denoted S_t , is ruled by a geometric Brownian motion whose average return and volatility are respectively noted m et σ . The relative variation of S_t is defined by the next equation:

$$\frac{dS_t}{S_t} = mdt + \sigma dW_t. \tag{2}$$

where W_t is a Brownian motion ($\sim N(0, \sqrt{t})$), defined as usual on a probability space $(\Omega, (\mathcal{F}_t)_t, P)$. In the sequel of this paper, F_t points out the total market value of

assets at time t . A fraction π_t of this amount is invested in risky assets whereas the remainder is placed at risk free rate. π_t is another parameter of control of our model. On the basis of equation. (2), we can formulate the dynamics of F_t , $\forall t \neq T_i$ $i = 1 \dots L$:

$$\begin{aligned} dF_t &= (1 - \pi_t)F_t r dt + \pi_t F_t dS_t \\ &= (r + (m - r)\pi_t)F_t dt + \pi_t \sigma F_t dW_t. \end{aligned} \quad (3)$$

The figure 2 presents the accounting balance sheet of the insurance company. The accounted equity is defined in our model as the difference between the total assets and the total mathematical provision.

| ASSETS | LIABILITIES |
|--------------------------------------|------------------------------|
| Risky asset $\pi_t F_t$ | Acc. Equity $n_t^1 R_t^1$ |
| Risk free asset $(1 - \pi_t) F_t$ | $n_t^i R_t^i$ |
| | $n_t^L R_t^L$ |

Figure 2: Balance sheet.

3. INSURER'S OBJECTIVE.

As mentioned earlier in the introduction, one considers that the insurance company tries to maximize both the utility drawn from the profit shares distribution, and from its wealth, at the end of a given horizon. Let us note T , the time horizon of the insurer. In order to avoid annoying discontinuities in our optimization model, we assume that:

$$T \leq \min\{T_1, T_2, \dots, T_L\}.$$

The wealth of the insurer at time T is here defined as the market value of the equity, which is here the difference between the market value of assets and the market or fair value of liabilities,

$$Equity = F_T - \sum_{i=1}^L n_T^i R_T^{r,i}.$$

Where $R_T^{r,i}$ is the market value, also called fair value, of one unit of account :

$$R_t^{r,i} = e^{-r(T_i - t)} C_i.$$

Note that the market value of equity plays an important role in the future Solvency II regulation. The solvency capital requirement must indeed be compared with the market value of equity (called Net Asset Value in Solvency II). That is for this reason that one

maximizes the expected utility of this quantity rather than the utility of the accounted equity. The economic utility of the equity is measured by a concave function noted $U_2(\cdot)$. The utility functions of profit shares are indexed by the number of contract: $U_{1i}(\cdot)$, $i = 1 \dots L$.

The value function $V(t, F_t, n_t^1, \dots, n_t^L)$ is defined as the maximal expectation of future utilities over the set of admissible controls:

$$V(t, F_t, n_t^1, \dots, n_t^L) = \max_{\pi_t, P_t^{i=1 \dots L}} \mathbb{E} \left(\int_t^T e^{-\rho(u-t)} \sum_{i=1}^L U_{1i}(P_u^i) du + e^{-\rho(T-t)} U_2 \left(F_T - \sum_{i=1}^L n_T^i R_T^{r,i} \right) \middle| \mathcal{F}_t \right), \quad (4)$$

where ρ is a rate pricing the time value of future utilities. We will see in the next section that the value function is solution of a partial differential equation, and that the optimal investment-profit shares policies may be inferred from it. There exists many classes of utility functions. In the sequel of this paper, one will focus on constant relative risk aversion (CRRA) utilities :

$$U_2(z) = u_2 \frac{z^\gamma}{\gamma} \quad U_{1i}(z) = u_{1i} \frac{z^\gamma}{\gamma} \quad i = 1 \dots L$$

where $\gamma < 1$ is the risk aversion parameter. The parameters $u_{1i=1 \dots L}$, u_2 are constant and allow us to discriminate the profit shares of contracts with various interest rate guarantees. Compared to other utility functions, working with CRRA functions has the advantage of rejecting strategies leading to a negative equity (given that $U_2(\cdot)$ is not defined for negative values).

4. HJB EQUATION.

From the stochastic control theory (see for e.g. Fleming & Rishel 1975), we know that the value function $V(t, F_t, n_t^1, \dots, n_t^L)$ is solution of a the Hamilton Jacobi Bellman (HJB) equation which is a PDE. Let us denote V_F , V_{FF} , V_{n^i} , V_t respectively the first order, the second order derivatives of $V(t, F_t, n_t^1, \dots, n_t^L)$ with respect to F_t and the first order derivative of $V(t, F_t, n_t^1, \dots, n_t^L)$ with respect to n_t^i and t . The HJB equation of our problem is:

$$0 = V_t - \rho V + \max_{\pi_t, P_t^{i=1 \dots L}} \left((r + (m-r)\pi_t) F_t V_F + \frac{1}{2} \pi_t^2 \sigma^2 F_t^2 V_{FF} + \sum_{i=1}^L \left(\frac{P_t^i}{R_t^i} V_{n^i} + U_{1i}(P_t^i) \right) \right), \quad (5)$$

under the terminal condition that

$$V(t, F_t, n_t^1, \dots, n_t^L) = u_2 \frac{\left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)^\gamma}{\gamma}$$

Differentiating the term maximized in the HJB equation, with respect to the variables of control, leads to the following optimal strategy:

$$\pi_t^* = -\frac{m-r}{\sigma^2} \frac{V_F}{V_{FF}} \frac{1}{F_t} \quad (6)$$

$$P_t^{i*} = U_{li}^{\prime-1} \left(-\frac{V_{n^i}}{R_t^i} \right) = \left(-\frac{V_{n^i}}{R_t^i} \frac{1}{u_{1,i}} \right)^{\frac{1}{\gamma-1}} \quad i=1 \dots L \quad (7)$$

The combination of equations (5), (6) and (7) gives the next PDE:

$$0 = V_t - \rho V + r F_t V_F - \frac{1}{2} \frac{(m-r)^2}{\sigma^2} \frac{V_F^2}{V_{FF}} + \sum_{i=1}^L u_{1,i}^{1-\gamma} \left(\frac{1}{\gamma} - 1 \right) \left(-\frac{V_{n^i}}{R_t^i} \right)^{\frac{\gamma}{\gamma-1}} \quad (8)$$

from we will guess $V(\cdot)$. One assumes that the value function depends on the market value of equity in the following way:

$$V(t, F_t, n_t^1, \dots, n_t^L) = b(t) \frac{\left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)^\gamma}{\gamma} \quad (9)$$

where $b(t)$ is an unknown function of time such that $b(T) = u_2$ (this ensures that the terminal condition is well satisfied). The partial derivatives of the value function are:

$$V_t = b'(t) \frac{\left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)^\gamma}{\gamma} - \sum_{i=1}^L b(t) \left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)^{\gamma-1} n_t^i r R_t^{r,i}$$

$$V_F = b(t) \left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)^{\gamma-1}$$

$$V_{n^i} = -b(t) \left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)^{\gamma-1} R_t^{r,i}$$

$$V_{FF} = b(t) (\gamma-1) \left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)^{\gamma-2}$$

If we insert those last partial derivatives into equation (8), one obtains an equation in which all terms are multiplied by $\frac{1}{\gamma} \left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)^\gamma$, a random quantity. From this observation, we infer that the function $b(t)$ must therefore be the solution of the following ordinary differential equation:

$$0 = b'(t) + b(t) \underbrace{\left(r\gamma - \rho - \frac{1}{2} \frac{(m-r)^2}{\sigma^2} \frac{\gamma}{\gamma-1} \right)}_{\phi} + (1-\gamma) b^{\frac{\gamma}{\gamma-1}} \underbrace{\sum_{i=1}^L u_{1,i}^{1-\gamma} \left(\frac{R_t^{r,i}}{R_t^i} \right)^{\frac{\gamma}{\gamma-1}}}_{D(t)} \quad (10)$$

under the terminal condition mentioned early $b(T) = u_2$. The solution of this Cauchy's equation is:

$$b(t)^{\frac{1}{1-\gamma}} e^{\frac{1}{1-\gamma} \phi t} = - \int_0^t D(s) e^{\frac{1}{1-\gamma} \phi s} ds + c^{st},$$

where

$$c^{st} = u_2^{\frac{1}{1-\gamma}} e^{\frac{1}{1-\gamma} \phi T} + \int_0^T D(s) e^{\frac{1}{1-\gamma} \phi s} ds.$$

Finally, after simplifications, one gets that the function $b(t)$ is defined as follows:

$$b(t) = \int_t^T D(s) e^{\frac{1}{1-\gamma} \phi (s-t)} ds + u_2^{\frac{1}{1-\gamma}} e^{\frac{1}{1-\gamma} \phi (T-t)}. \quad (11)$$

The shape of $b(t)$ will be presented in the example developed in the next section. The value function being solely determined by equations (9) and (11), one can now calculate the optimal investment and profit sharing policy by equations (6) and (7) :

$$\pi_t^* = \frac{m-r}{\sigma^2} \frac{1}{1-\gamma} \frac{\left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right)}{F_t} \quad (12)$$

$$P_t^{i*} = b(t)^{\frac{1}{\gamma-1}} \left(u_{1,i} \frac{R_t^i}{R_t^{r,i}} \right)^{\frac{1}{1-\gamma}} \left(F_t - \sum_{i=1}^L n_t^i R_t^{r,i} \right) \quad i=1 \dots L \quad (13)$$

As illustrated in figure 2, the optimal amount of stocks that the company should hold is equal to the equity times a multiplier. This multiplier is itself the product of the cost of risk $\frac{m-r}{\sigma^2}$ and of a risk aversion $\frac{1}{1-\gamma}$ coefficient that tends to $+\infty$ when γ is close of 1

(i.e. when the insurer is nearly insensitive to risk).

| ASSETS | | | LIABILITIES | |
|---|--|---|-------------------|--|
| Risky asset : Equity x multiplier | | Multiplier : $\frac{m-r}{\sigma^2} \frac{1}{1-\gamma}$ | Mkt Equity | |
| | | | $n_t^1 R_t^{r,1}$ | |
| Risk free asset | | | $n_t^i R_t^{r,i}$ | |
| | | | $n_t^L R_t^{r,L}$ | |

Figure 3: Balance sheet.

The formulas (13) of optimal profit shares are less intuitive. However, one sees that the optimal profit share granted to one contract is a fraction of the equity. This fraction is the product of a coefficient identical for all contracts $b(t)^{\frac{1}{\gamma-1}}$ and of a weight $\left(u_{1,i} \frac{R_t^i}{R_t^{r,i}}\right)^{\frac{1}{1-\gamma}}$. This weight is bigger than $(u_{1,i})^{\frac{1}{1-\gamma}}$ if the mathematical provisions is bigger than its market value, and is smaller than $(u_{1,i})^{\frac{1}{1-\gamma}}$ otherwise.

5. EXAMPLE.

This section illustrates numerically our results. Let us consider a portfolio of two participating life policies, both of maturity $T_1 = T_2 = 10$ years. Policyholders purchase a contract that delivers 1000 Euros at maturity but choose respectively a guarantee $r_1^g = 2\%$ and a guarantee $r_2^g = 0\%$. The initial numbers of units of account are set to one, $n_0^1 = n_1^2 = 1$, while the capitals at maturity are set to thousand, $C_1 = C_2 = 1000$. The mathematical provisions at time $t = 0$ (which are equal to premiums paid at $t = 0$) are therefore:

$$\begin{aligned} R_{t=0}^1 &= 1000.e^{-2\%.10} = 819 \\ R_{t=0}^2 &= 1000.e^{-0\%.10} = 1000 \end{aligned}$$

The accounting equity is equal to 3% of the mathematical provisions. So the initial value of the total assets is worth

$$F_{t=0} = (1 + 3\%)(R_{t=0}^1 + R_{t=0}^2) = 1874$$

The risk free rate is $r = 3\%$. The mean and standard deviation of the risky asset are respectively $m = 6\%$, $\sigma = 25\%$. The fair value of liabilities is then:

$$R_{t=0}^L = 1000.e^{-3\%.10} = 740$$

$$R_{t=0}^2 = 1000.e^{-3\%.10} = 740$$

The insurer's risk aversion parameter and time horizon are respectively set to $\gamma = 0.2$ and $T = 10$. The time value of future utility is worth $\rho = 1\%$. The policyholder choosing a guarantee of 0% expects to obtain a capital at term higher than the customer opting for a higher guarantee. In order to reflect this in the profit sharing policy, one chooses the following weights: $u_{11} = (r - r_1^g)/r = 0.33$, $u_{12} = (r - r_2^g)/r = 1$ and $u_2 = 1$. The optimal investment and profit sharing strategies have been discretized with of step of time equal to $\Delta t = 0.1$.

Two scenarios are studied. In the first one, the return of the risky asset is stochastic. In the second scenario, the return of this asset class is constant and is worth 4.5%. The figures 3 and 4 respectively depict the evolution of assets and provisions in the stochastic and deterministic scenarios. In both cases, the accounting value of provisions converges towards the market value of provisions. This convergence comes from the distribution of profit shares along the lifetime of policies. One also notes that in the stochastic scenario, despite the random growth of assets, the growth of provisions remains smooth.

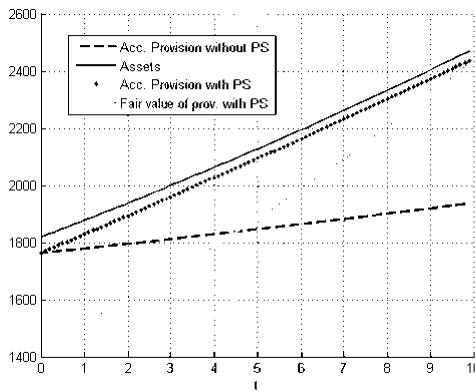


Figure 4: Evolution of assets and provisions. Deterministic scenario.

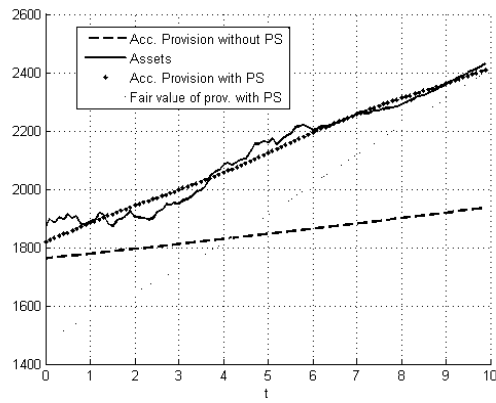


Figure 5: Evolution of assets and provisions. Stochastic scenario.

The graph 5 reveals that the optimal amount of risky asset decreases with time. If we remember equation (12), the optimal fraction of the asset invested in stocks is directly proportional to the equity level. When the position in stocks decreases, so does the equity. Our insight is confirmed by figure 6 presenting the equity (accounted and in market value) as a percentage of the total asset F_t . One also observes a convergence of the accounted equity toward the market value of the equity. This convergence is the result of the profit sharing mechanism that redistributes partly the wealth to policyholders.

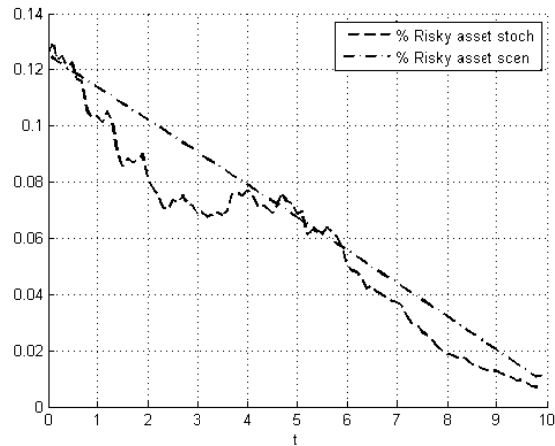


Figure 6: Investment policy, % of F_t .

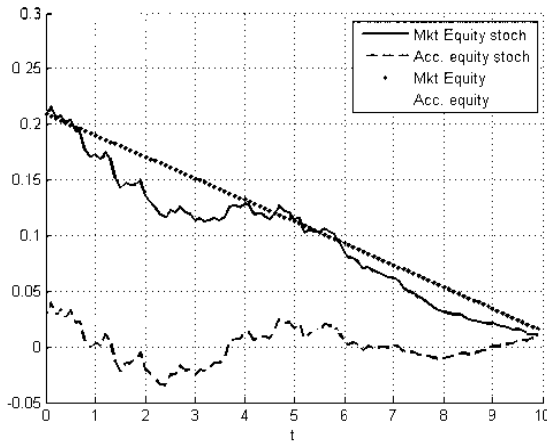


Figure 7: Accounted and Market Equity as % of F_t .

Figure 7 presents the instantaneous return granted to the total mathematical provision. One has defined this return as follows

$$return = \log \left(\frac{n_{t+\Delta t}^i R_{t+\Delta t}^i}{n_t^i R_t^i} \right) \frac{1}{\Delta t}$$

If the number of units of account does not rise during the interval of time Δt , the instantaneous return is equal to the guaranteed interest rate. We remark that the insured having chosen a lower guarantee receives a higher return. This observation confirms an interesting feature of our approach: by an adapted choice of weights $u_{1,i}$, one can discriminate profit shares between policies with different guarantees.

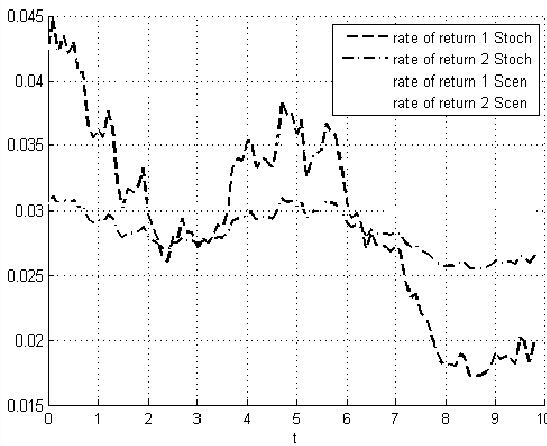


Figure 8: Instantaneous return.

We end up this section by a small analysis of the function $b(t)$ involved in the calculation of the optimal profit share. The graph 8 shows that $b(t)$ is a decreasing function of time, highly sensitive to the risk aversion level γ and to the risk free rate r . The function is clearly less sensitive to changes of return m and volatility σ of stocks.

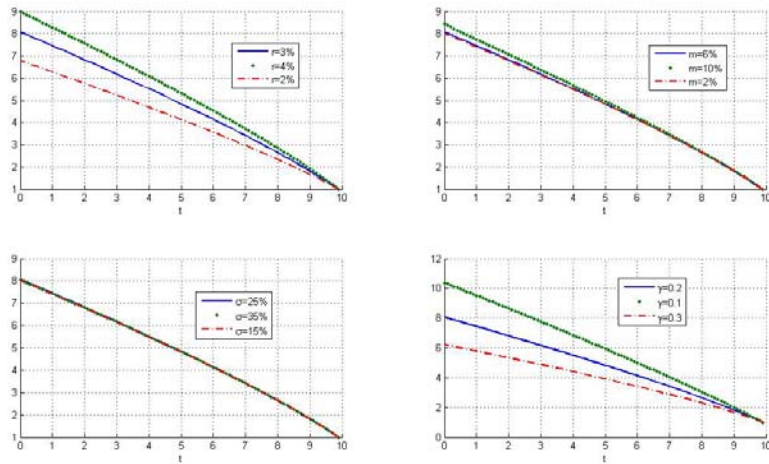


Figure 9: Analysis of $b(t)$

6. CONCLUSION.

This paper proposes a model to optimize both the investment and profit sharing policies. We have favored an economic approach rather than a method based on option pricing. This approach is particularly well adapted to countries where the regulator let insurers choose freely the level of profit sharing, and where finally the only constraints to distribute or not a bonus are the competition between companies and the shareholders' waitings. The maximization of utilities drawn from future profit shares and from the terminal wealth allows to justify the allocation of profit shares between contracts having different characteristics and is useful to manage the insurance company on a long term horizon. However, there remains many points that should be investigated. A first one, is to study the solution when the insurer's time horizon is longer than the maturity of contracts. This introduces annoying discontinuities in the solution. In future research, one could also try to add stochastic interest rates or a solvency constraint on the terminal wealth.

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