DECISIONAL OPTIMALITY IN LIFE REINSURANCE MODELING

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Résumé:

L’actualité économique et réglementaire des entreprises d’assurance vie - épargne et prévoyance - intensifient la réflexion de l’arbitrage entre rentabilité et risque/capital. Dans ce contexte, la réassurance se révèle d’une utilité toute particulière en présentant une alternative (en sus de son rôle traditionnel) tant à l’augmentation des rendements par ‘accroissement de la prise de risque qu’au financement des exigences en capital via les marchés financiers.

Nous cherchons, dans le cas d’un portefeuille de prévoyance, à observer les effets de la réassurance sur diverses mesures de risque et de rentabilité connues, et à en optimiser les niveaux pour des contraintes de structure et réglementaires. Cela nous conduit à définir précisément une structure de réassurance, les paramètres que nous faisons varier, puis leurs effets sur nos indicateurs afin d’obtenir les frontières efficientes pour chaque combinaison de mesures de rentabilité et de risque.

Mots-clés : Réassurance vie, mesure de risque, solvabilité, exigence en capital, frontière efficiente, optimalité

Abstract

More than ever life insurance companies (individual & group covers, savings) face real thinking of arbitrage between reward and risk/capital, due to economic turmoil and regulation. Reinsurance appears very useful as it presents an efficient alternative to increasing reward by increasing the risk, but also as an alternative to the funding (for solvency needs) from the capital markets.

We aim at observing, for an individual cover portfolio, the effects of the reinsurance on several risk measures and several reward measures, and thus being able to optimize the

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reinsurance levels under structure and regulatory constraints. This leads us to precisely define a reinsurance structure and the parameters supposed to vary, and their effects on monitored indicators to obtain efficient frontiers for each combination of risk/reward measures.

**Key words:** Life Reinsurance, risk measure, solvency, capital requirement, efficient frontier, optimality

1. **ACADEMIC BACKGROUND AND MOTIVATIONS**

Reinsurance market is traditionally dedicated to risk transfer. This transfer enables the insurance companies to be protected against peak, frequency and catastrophic risks. But the current economic context and the regulation make this market to come with innovative solutions. Indeed, solvency regulation requires better knowledge and better management of risks and resources in front of these risks. Reward measure, capital management and investment, minimum own funds requirements, ORSA, IFRS 4, ... a context in which arise new strategies and management for life and non-life insurance companies, and in which reinsurance structuring takes a wider role.

On the one hand, academic research has for long defined many issues and solutions to optimize risk transfers, to begin with heavy-tailed distributions and extreme risks protections, especially with ruin theory. This constitutes a first current in which papers agree to promote stop loss reinsurance as optimal. Borch (1969) proved the stop-loss reinsurance to be optimal in terms of variance reduction, Ohlin (1969) extended the results for any continuous dispersion measures, then Lemaire (1973) with the spread measure who also derive the optimal level for the stop loss reinsurance. Cai, Tan, Weng and Zhang continued this analysis and established that either stop-loss, quota-share or a combination of both is optimal for the Value-At-Risk (and TVaR) depending on the position of the loss distribution to the expected premium. This let us observe two main features on which non-proportional structures can be challenged : their price and their perimeter. The price induces that there are market conditions on the acceptance of a structure, and that the limited number of agents (especially in life market) make the no-arbitrage condition difficult to assess. We mean through this that we consider a reinsurance structure as optimal if first it can be accepted by some reinsurers. Let's give some details here. One postulate of this current is that we can have the same net premium for different structure. This implies that price is commonly accepted to be given with a fixed measure, and that optimality results
depend on the choice of this pricing measure. It implies that optimization on structures is reduced to an optimization on price of these structures. It turns out that the specificity of risks transferred on the life market (long durations, products linked to assets, multiple disability states, commercial network, ...) encourage reinsurers to enlarge their risk appreciation, integrating charge for capital tied up, commissions to commercial networks, or financing approach in their pricing - such reasons that limit the expectation principle in life reinsurance. Second issue is about the perimeter of the reinsurance, that defines the underlying random variable on which measures are optimized : they are often representing a loss distribution but of one portfolio ? one line of business or segment ? the whole cedant business ? For each perimeter exist different stop loss structures, that in principle work the same way but require different level of information, analysis and surely pricing, depending as well on the number of accounting items comprised in the stop-loss treaty. A reinsurer analysing a stop loss on the whole income statement (i.e. all accounting items taken into account) will certainly ask for top-level business plan, asset allocation, profit distribution, reserving methods, expenses allocations ... and any source of risk can influence the price. This also highlights new limits to the issue : the availability of the information, whether for the cedant (concept of "total information") or the reinsurers (concept of "asymmetrical information").

We point out here that these optimizations have moved from an optimization on structures to an optimization on price of top-level non-proportional reinsurance : this criticism of the classical approach appears in recent studies (Aon Benfield (2015), Albrecher and Haas (2012) and Egoshina et al. (2013), and are largely motivated by practitioners (reinsurance brokers, consulting firms, reinsurers).

On the other hand, we can state that another trend of research adopts a financial approach with mean and variance as fixed measures to be optimized, and the focus is on the structure forms and levels. de Finetti (1940) (precisions in Pressaco et al. (2010) and later Markowitz (1952) introduced the famous risk-return optimization under the mean-variance program. Through this analogy in insurance, each asset is represented by a policy with an expected return (the premium) and an expected loss (capital or annuity to pay). More recently, Walhin, Glineur and Lampaeart revisited the de Finetti method with convex optimization, and gave illustrated results on proportional reinsurance optimal levels. This analysis enables to get the individual optimal cession rates for each risk (or group of risk). Following the fundamentals works of de Finetti, Pressacco, Serafini and Ziani (Pressaco et
al. (2010) and Pressaco and Serafini (2007)) completed the research giving closed formulas when group correlation is added to the quadratic mean variance program. Under these very same measures, Verlaak and Beirlant (2002) study the cardinality and layering of reinsurance (i.e. the order and split of reinsurance structures in a reinsurance program) for combinations of the main reinsurance structures. They are also confronted to the price issue that they solve using the expected principle with loading on the standard deviation of retained risk, and we could make the same comments as above relatively to this hypothesis for life reinsurance.

Complementary papers such as Lampaert and Walhin (2005) or Plantin (1999) introduce the effects of reinsurance on capital indicators and on insurer's balance sheet, where it is shown that financing can come from a reduction of the liabilities. These approaches precede the current assessment of reinsurance on capital and the need to quantify it thanks to simulation, with for example the counterparty risk introduced in Solvency 2 as a function of reinsurers' aggregated liabilities and ratings.

Recently, Albrecher and Haas (2012) develop an approach that consists in deriving a retained loss function transforming heavy-tail distribution in an "exponentially bounded tail" distribution, replacing the standard truncation with excess of loss structures, and hence reducing the reinsurance premium for this cover but allowing risk transfer. However, it is admitted in research report (Egoshina et al. (2013)) that application of the alternative reinsurance structure may be limited for life portfolios which already have a light-tail distribution due to the convergence of losses to normal distribution.

This being said, we do not criticize the powerful and technical analytical results, but we rather tend to challenge the definition of the optimal programs through the level of complexity of reinsurance contracts as well as the objective functions. Firstly, we define a reinsurance structure by the combination of the following features:

- perimeter
- trigger event (non-cat or cat: cat covers need to be specified an event that triggers the payment from the reinsurers)
- form (proportional or non proportional, per head (per risk) or aggregate)
- level parameters (retention level, cession rate, limit)
- commercial parameters (price for non prop., commission rate, profit share rate)
- duration.
Secondly, our study consists in modeling and observing the variations of chosen indicators induced by the variation of each feature here above. Let's note that features like duration or profit sharing (that are life reinsurance market specificities) have rarely been included in optimization studies.

Thus, the classical mono-objective program (mean-variance or mean-VaR) is transformed into a complete and transverse multi-objective optimization with endogenous and exogenous constraints: maximization of multi-year net profit, ceded deviation and peak risk, cession in capital required under diverse regulation, and by means of existing reinsurance structures and market-based conditions. This is where our approach is motivated and innovative: confronting risk indicators, market and regulation, and combining financial risk-reward approach, appropriate life line-by-line actuarial modeling and Solvency 2 standard-formula calculations.

The purpose of this paper is to prove the existence of price-independent efficient frontiers for reinsurance structures on a life portfolio, and that the shape is unique for each combination of risk/reward measures.

The determination of these efficient frontiers makes this analysis a market and consistent decisional tool for reinsurance buyers regarding real-world arbitrage between cost, risk and capital management.

1.1 Overview

In Section 2, we derive the optimization programs with the standard life insurance notations, with a reminder on reinsurance structures and corresponding cashflows. Results are expressed as functions of reinsurance parameters (form, levels, duration, ...).

In Section 3, we solve the optimization programs with analytical tools using Markowitz and de Finetti methods. We underline the limits of these analytical solutions.

In Section 4, we use Monte-carlo and line-by-line trials to simulate gross and net cashflows in order to monitor any statistic on distributions. These statistics enable us to plot the empirical efficient frontiers. This conducts us to focus on duration and price of structures to observe the sensitivity of cashflows.

In Section 5, we finally derive the functions that characterize transformations between gross and net random variables, and that will enable to derive direct relations between standard deviation, VaR and RORAC independently from the reinsurance parameters and hence obtaining the equations of the empirical efficient frontiers.
In Section 6, a conclusion sums up the analytical then simulating approaches that lead to optimizing life reinsurance.

Note that notations of best estimate liabilities are derived from Planchet, Guibert and Juillard (2010) and details on calculations can be found in Saunier (2013).

2. FORMAL APPROACH OF REINSURANCE ON DEATH RISK

2.1 Features of the insurance product and gross cashflows (before reinsurance)

The product we considered is very simple: the insured pays an annual premium and the insurer is engaged to pay a defined amount in case of death of the insured during the year. The premium is determined by a table of premium depending of the age of the insured. It is revised annually in our case. In our study, we also consider no reserve, as the product is on annual basis. But this hypothesis does not change the results and we could take them into account with heavier notations in the claims.

From now on, "gross" means before reinsurance and "net" after all reinsurance, the structure being implicitly considered. We introduce the following notations:

- $n$ number of policies in the portfolio
- $K_i$ sum assured per policy $i$
- $T_{xi}$ date of event (death) for $i$
- $T_i$ end date of contract $i$
- $q_{xi}$ death rate at age $x_i$ of the insured $i$
- $w_{xi}$ lapse rate at age $x_i$ of the insured $i$
- $S(t)$ total amount of claim for the period $t$
- $P(t)$ total premium earned for the period $t$
- $Res(t)$ technical result for the period $t$
- $BEL$ "Best Estimate Liabilities"
- $\delta(t)$ discount factor at end of the period $t$

We consider that the following formal projections are made period by period, with calculation at the end of the year. Premiums are supposed to be paid at the beginning of the year and claims at year end. It is a common hypothesis that could bias the results but simplifies the formulas. In the simulation (second part), the real dates of payment are taken into account. The one-year technical result before reinsurance can be written:

$$Res(1) = \sum_{i=1}^{n} P_i(1) - S_i(t) \cdot 1_{T_i \leq 1}$$
Results for dates $t=2,3, \ldots$ can be obtained in the same way. No new business is taken into account to lighten the formalization and simulations. Besides, we note that $S_t(t) = K_t$, i.e. each loss is a total loss.

### 2.2 Best estimate liabilities before reinsurance

Under Solvency 2, (cf. EIOPA (2010)), best estimate liabilities (as a random variable) for the cedant is the net present value of future cashflows:

$$BEL = VA(FutureClaims - FuturePremiums)$$

with a constant discount factor over the year and without general costs:

$$BEL = \sum_{t=1}^{\infty} \sum_{i=1}^{n} \delta(t) \cdot \left(K_i \cdot 1_{t-1<\tau_{x_i}<t} - P_t(t) \cdot 1_{\tau_{x_i}\geq t-1}\right)$$

Taking the mean of this BEL gives the Solvency best estimate liabilities:

$$E(BEL) = \sum_{t=1}^{\infty} \sum_{i=1}^{n} \delta(t) \cdot \left(K_i \cdot 1_{t-1} p_{x_i} \cdot q_{x_i+t-1} - P_t(t) \cdot 1_{t-1} p_{x_i}\right)$$

Let’s notice that $1_{t-1} p_{x_i}$ represent the combined rate of survival to death and lapse. $p_{x_i} = (1 - q_{x_i})(1 - w_{x_i})$ with notations above.

The prudential balance sheet can be written:

<table>
<thead>
<tr>
<th>$t=0$</th>
<th>Fair Value Assets</th>
<th>Initial Own Fund</th>
<th>Best Estimate Liabilities = $E(BEL)$</th>
</tr>
</thead>
</table>

We won’t detail here the calculation steps to get the expression of Net Asset Value for each shock, and rather go through the reinsurance impact on all expressions and balance sheet above.

### 2.3 Quick reminder on reinsurance structures

In the following we will mainly consider proportional reinsurance, but we will also see how non proportional structures can be implemented in our analysis.

Among proportional structures, we distinguish quota-share and surplus: the quota-share will have a retention of 0 and a limit (subscription limit = "plein de souscription"), where the retention of the surplus will necessarily be over 0. The cession rate of the surplus is traditionally at 100% (for the sum assured over the retention) where for a quota-share it can be comprised anywhere between 0% and 100%. Through these structures, same proportion is ceded in terms of premiums and losses. Prior calculations have then to be computed in function of retentions, limits, cession rates. We will write these calculations in
the following paragraphs. After the effect of reinsurance on premiums and losses, proportional reinsurance can be featured with profit sharing clauses, or profit participation: at the end of the life of the treaty (it can be from 1 year to 1 generation), the reinsurer pays back a certain amount of the reinsurance profit if it has been positive.

For the non proportional reinsurance, the losses can be ceded per risk (per head in case of life reinsurance) or on an aggregate basis. In both cases, the premium paid by the cedant is the result of a risk analysis and negotiation with reinsurer(s). Given the fact that the premium (usually a rate) is fixed (except if revision clauses) during the treaty lifetime, this kind of reinsurance is mainly short term (1 to 3 years) as as reinsurers’ risk appreciation may evolve over the time with the risk factors (historical loss data, portfolio size increase, change in perimeter reinsured, higher capital requirement for reinsurer, ...). 

In a first approach, we will develop the optimization of proportional reinsurance, due to our motivations in Section 1 on the non-proportional pricing. A paragraph will deal further in the study about the non-proportional reinsurance and how results can be easily adapted and above all compared from the ones obtained with proportional reinsurance.

2.4 Cashflows net of reinsurance

We can at this point define the technical account for the reinsurance treaty:

<table>
<thead>
<tr>
<th>Debit</th>
<th>Credit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ceded premiums</td>
<td>Ceded losses</td>
</tr>
<tr>
<td>reinsurance fees</td>
<td>Commissions</td>
</tr>
<tr>
<td></td>
<td>Profit participation</td>
</tr>
</tbody>
</table>

If needed (not in our example to simplify the expressions), we can also add in this technical account the variation of ceded reserves.

We begin the study by considering an existing structure of proportional reinsurance - a quota share and a surplus. For this structure, let’s have the following variables:

- $\beta$ quota share cession rate
- $\beta_i$ quota share cession rate adjusted to retention, by insured $i$
- $\gamma_i$ surplus cession rate, by insured $i$
- $Ret, Lim$ retention and limit of surplus
- $t_{PP}^{QS}$ et $t_{PP}^{SP}$ profit participation (PP) rates for quota share (QS) and surplus (SP)

Step by step, let’s detail the cashflows after different combinations of quota share and surplus.

With a quota share, the amounts net of reinsurance are:
Given a limit $\text{Lim}$ on sum assured on the quota share, the individual cession rate becomes:

$$\beta_i = \beta \cdot \min\left(1, \frac{\text{Lim}}{K_i}\right)$$

The technical result is finally expressed:

$$\text{Res}^\text{net}(1) = \sum_{i=1}^{n} (1 - \beta_i) \cdot \left(P_i(1) - K_i \cdot 1_{T_i \leq 1}\right)$$

If we simplify with $\text{Res}_i = P_i(1) - K_i \cdot 1_{T_i \leq 1}$ (the individual technical result) then:

$$\text{Res}^\text{net}(1) = \sum_{i=1}^{n} (1 - \beta_i) \cdot \text{Res}_i(1)$$

We can now add the profit participation that gives net result:

$$\text{Res}^\text{net}(1) = \sum_{i=1}^{n} (1 - \beta_i) \cdot \text{Res}_i(1) + t_{\text{pp}} \cdot \left(\sum_{i=1}^{n} \beta_i \cdot \text{Res}_i(1)\right)^+$$

This expression can be completed (and complicated) when we introduce fees of the reinsurer (on the ceded premiums) and commissions (calculated on gross premiums). The result is no longer linear.

Now let’s get to the surplus: the amount of losses ceded in the surplus is equal to:

$$\min(\max(K_i - \text{Ret}; 0), \text{Lim} - \text{Ret})$$

that enables us to express the individual cession rate in the surplus:

$$\gamma_i = \min\left(\max\left(1 - \frac{\text{Ret}}{K_i}; 0\right), \frac{\text{Lim} - \text{Ret}}{K_i}\right)$$

Identically to quota share, the result net of surplus with profit participation is:

$$\text{Res}^\text{net}(1) = \sum_{i=1}^{n} (1 - \gamma_i) \cdot \text{Res}_i(1) + t_{\text{pp}} \cdot \left(\sum_{i=1}^{n} \gamma_i \cdot \text{Res}_i(1)\right)^+$$

where $\text{Res}_i(\cdot)$ remains the same individual technical result defined above.

From this point, the reinsurance is made up with a quota-share with profit participation, a surplus with profit participation and the last surplus is not with profit participation. The reinsurance layers are continuous: the retention of the upper level is the limit of the lower one. With these reinsurance layers, we now obtain the final expression of the result net of reinsurance:
\[ Res^{net}(1) = \sum_{i=1}^{n} Res_i(1) \cdot \left(1 - \beta_i - \gamma_{1,i} - \gamma_{2,i}\right) \]

\[ + t_{PP}^{OP} \cdot \left(\sum_{i=1}^{n} \beta_i \cdot Res_i(1)\right)^+ \]

\[ + t_{PP}^{OP} \cdot \left(\sum_{i=1}^{n} \gamma_{1,i} \cdot Res_i(1)\right)^+ \]

with

\[ \beta_i = \beta \cdot \min\left(1; \frac{Ret1}{K_i}\right) \]

\[ \gamma_{1,i} = \min\left(\max\left(1 - \frac{Ret1}{K_i}; 0\right); \frac{Lim1 - Ret1}{K_i}\right) \]

\[ \gamma_{2,i} = \min\left(\max\left(1 - \frac{Lim1}{K_i}; 0\right); \frac{Lim2 - Lim1}{K_i}\right) \]

This being established, let's map the way reinsurance is taken into account in regulatory environment.

### 2.5 Reinsurance in Solvency 1 & 2

Calculation of capital requirement in Solvency 1 is quite direct: in the case of death risk, it is calculated on the total exposure. If we consider a reinsured product with a remaining duration of more than 5 years, the amount of gross or net capital requirement is 0.3% of the gross or net exposure. The cession (\(gross - net\)) is limited in its effects on the margin to 50% as mentioned in the "Code des Assurances" (2013) article R334-13).

As regards Solvency 2, we can build the prudential balance sheet. Let's notice that we do not consider effect of reinsurance on assets, to simplify the analysis on this term- assurance product. Such effects of reinsurance can still be quantified and exist.

Before reinsurance:

\[ t = 0 \]

\[ \begin{array}{c|c}
\text{Assets} & \text{Free surplus} \\
& + \text{SCR} \\
& + \text{Risk margin} \\
& \text{Best Estimate liabilities}
\end{array} \]

In this balance sheet, the reinsurance effect is very specific: according to the boundaries contract clause, the duration of the reinsurance to be taken into account in calculations of best estimate recoveries must be equal to the duration given in the existing
reinsurance treaty (EIOPA 2010) TP.2.12 - The calculation of the best estimate should only include future cash-flows associated with existing insurance and reinsurance contracts).

With reinsurance, the balance sheet above is transformed into:

$$t = 0$$

<table>
<thead>
<tr>
<th>Assets</th>
<th>Free surplus</th>
<th>Best Estimate liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+ SCR</td>
<td>Ceded premiums</td>
</tr>
<tr>
<td></td>
<td>+ Risk margin</td>
<td></td>
</tr>
<tr>
<td>Best Estimate recoveries</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

which gives with the correct notations, with a quota-share structure for example:

$$t = 0$$

<table>
<thead>
<tr>
<th>$$A_0$$</th>
<th>Free surplus</th>
<th>$$P_i(1)$$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+ SCR</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+ RM</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$$E(BE)$$</td>
<td></td>
</tr>
</tbody>
</table>

$$\beta \sum_{i=1}^{n} K_i \cdot q_{x_i}$$

It is difficult to classify the profit sharing (or profit participation) cashflow. Indeed, it is conditioned to the reinsurance technical result and paid after all the other cash flows (in accounting statements). Being asymmetrical, we will not take it into account in the best estimate liabilities. Still it is a cashflow due by the reinsurer, so it will be counted in the default risk of the counterparty.

With the balance sheet above and the Solvency 2 methods applying shocks, and after computations no detailed here, we obtain the SCR sub-modules:

given that $$CF_i(t) = K_i \cdot p_{x_i} \cdot q_{x_i+t-1} - P_i(t) \cdot p_{x_i}$$, SCR for mortality or lapse shock is equal to:

$$SCR_{mortality/lapse} = \sum_{t=1}^{\infty} \delta(t) \sum_{i=1}^{n} \left(1 - (\beta_i - \gamma_{1,i} - \gamma_{2,i}) \cdot 1_{t \leq \text{exp}}\right) \cdot (CF_i^{\text{act}}(t) - CF_i(t))$$

As regards the other SCR submodules, $$SCR_{\text{expense}}$$ is quite direct to compute from the central best estimates liabilities. For the SCR Catastrophic, formula above remains valid if we compute the shock only on first year, inducing a decrease in premiums on following years (as more death on the first year).
Let’s give more details on the default risk and the calculation of $SCR_{counterparty}$. All reinsurance cashflows are subject reinsurer's ability to pay. To quantify this risk, Solvency 2 introduces the $SCR_{default}$ module (QIS5 SCR. 6.13 in EIOPA (2010)). With the table of probabilities and the formula of LGD given hereunder, the cedant has to quantify for each reinsurer (some simplifications are available) the amount of recoverables at stake and the impact on $SCR_{underwriting}$ (through the risk mitigating effect).

<table>
<thead>
<tr>
<th>Rating</th>
<th>Default Probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.002%</td>
</tr>
<tr>
<td>AA</td>
<td>0.01%</td>
</tr>
<tr>
<td>A</td>
<td>0.05%</td>
</tr>
<tr>
<td>BBB</td>
<td>0.24%</td>
</tr>
<tr>
<td>BB</td>
<td>1.20%</td>
</tr>
<tr>
<td>Not rated</td>
<td>4.175%</td>
</tr>
<tr>
<td>CCC</td>
<td>4.175%</td>
</tr>
</tbody>
</table>

*Table 1: QIS5 S&P Default probabilities*

Formula of *Loss-given default*

$$LGD_j = (1 - RR) \cdot (\text{Recoverables}_j + RM_j - \text{Collaterals}_j)$$

where $RR$ is the recovery rate, $RM$ is the risk mitigating effect which represents the amount of $SCR_{underwriting}$ transferred to each reinsurer: if a reinsurer is in default, firstly the cedant does not get the recoverables, but then the absence of these recoverables must be taken into account in the best estimate liabilities. The cedant keeps more risk hence must tie up more required capital than before the reinsurer's default.

### 3. OPTIMIZATION PROGRAMS - EXISTING SOLUTIONS AND LIMITS

Our study aims at studying reinsurance conditions satisfying risk appetite or risk aversion to maximize cedant's utility function. We need then to determine and introduce risk and reward indicators, such as net mean result, net standard deviation, Value-at-Risk, minimum and range of distribution, S1 capital and SCR.

On a first intuition, we start with Markowitz resolution on mean variance program with equality constraints. Let's consider proportional reinsurance on a portfolio of $n$ policies. The reinsurance defines a cession rate $a_i$ for each policy $i$. The Markowitz program can be written,
\[ \min A' \Omega A \]
\[ \text{under} (U - A)'E = E_{obj} \]
\[ \text{under constraint on } A \]

with

\[
A = \begin{pmatrix}
1 - a_1 \\
\vdots \\
1 - a_n
\end{pmatrix}
\]
\[
E = \begin{pmatrix}
E(\text{Res}_1) \\
\vdots \\
E(\text{Res}_n)
\end{pmatrix}
\]
\[
\Omega = \begin{pmatrix}
\text{Var}(\text{Res}_1) & 0 & \cdots & 0 \\
0 & \ddots & \cdots & \vdots \\
0 & \cdots & \ddots & 0 \\
0 & \cdots & 0 & \text{Var}(\text{Res}_n)
\end{pmatrix}
\]

We won't detail the lagrangian computations here but as a result, we observe that
- without constraint on \( A \), the solution range is a straight line.
- either we need \( n \) constraints for each cession rate
- or we define a single linear condition on \( A \).

In the two last cases, either \( n \) equalities are heavy and impossible to define or one single is not enough and the results are highly dependent and not always defined according to the condition.

As a final result of this first approach, we realize that mean-variance optimization under linear constraints is not enough. We then need to look in de Finetti's research to introduce \( n \) inequalities on \( A \). The optimization program on minimal variance subject to objective result is expressed as follows (where \( a_i \) represents policy \( i \) cession rate):

\[
\min \text{Var}(r_{\text{net}}) = \sum_{i=1}^{n} (1 - a_i)^2 \text{Var}(S_i)
\]
\[ \text{under } \forall i, a_i \leq 1 \]
\[ \text{under } \forall i, a_i \geq 0 \]
\[ \text{under } \sum_{i=1}^{n} (1 - a_i)(P_i - E(S_i)) = res_{obj} \]

By using convex optimization results, we can derive the lagrangian (every objective and constraint function is convex):
\[
L = \sum_{i=1}^{n} ((1 - a_i)^2 \text{Var}(S_i) + x_i(a_i - 1) - y_i a_i) + \lambda \left( \sum_{i=1}^{n} a_i(P_i - E(S_i)) - \sum_{i=1}^{n} (P_i - E(S_i)) + \text{res}_\text{obj} \right)
\]

thus the Karush-Kuhn-Tucker (KKT) conditions prove the existence of an optimal point and impose

\[-2\text{Var}(S_i)(1 - a_i) + x_i - y_i + \lambda(P_i - E(S_i)) = 0 \quad \forall i = 1 \ldots n \quad (1)\]

\[a_i \leq 1 \quad \forall i = 1 \ldots n \quad (2)\]

\[a_i \geq 0 \quad \forall i = 1 \ldots n \quad (3)\]

\[x_i \geq 0 \quad \forall i = 1 \ldots n \quad (4)\]

\[y_i \geq 0 \quad \forall i = 1 \ldots n \quad (5)\]

\[x_i(1 - a_i) = 0 \quad \forall i = 1 \ldots n \quad (6)\]

\[y_i a_i = 0 \quad \forall i = 1 \ldots n \quad (7)\]

\[\sum_{i=1}^{n} a_i(P_i - E(S_i)) = \sum_{i=1}^{n} (P_i - E(S_i)) - \text{res}_\text{obj} \quad (8)\]

3 cases appear :

1) \(x_i > 0\). So (6) gives \(a_i = 1\), so by (7) \(y_i = 0\).

In (1), this gives : \(x_i = -\lambda(P_i - E(S_i))\).

Condition (4) finally imposes

\(\lambda(P_i - E(S_i)) < 0\)

2) \(y_i > 0\). Then (7) gives \(a_i = 0\), so by (6) \(x_i = 0\).

In (1), this gives : \(-2\text{Var}(S_i) - y_i + \lambda(P_i - E(S_i)) = 0\).

Condition (5) finally imposes \(-2\text{Var}(S_i) + \lambda(P_i - E(S_i)) > 0\)

or

\[\frac{\lambda(P_i - E(S_i))}{2\text{Var}(S_i)} > 1\]

3) last possible combination is \(x_i = 0\) and \(y_i = 0\).

then, \(-2\text{Var}(S_i)(1 - a_i) + \lambda(P_i - E(S_i)) = 0\) or :

\[a_i = \frac{1}{\frac{\lambda(P_i - E(S_i))}{2\text{Var}(S_i)}}\]

this is acceptable according to (2) and (3) when :

\[0 \leq \frac{\lambda(P_i - E(S_i))}{2\text{Var}(S_i)} \leq 1\]

We can sum up these 3 cases and give \(a_i\) : if
\[ \phi_i = \frac{p_i - E(S_i)}{2\text{Var}(S_i)} \]

we have

\[ a_i = 1 - \min(1; \max(0; \lambda \phi_i)) \]

Now, we insert the obtained expression for the \( a_i \) in (8) and we can obtain the value of \( \lambda \) by numerical solver. To sum up our analytical work, we found optimal individual cession rates that minimize global variance under constraint of global mean result: this is the de Finetti efficient frontier. We will plot this frontier in following paragraph.

Before this, analytical work can be pursued: now we've observed the efficient frontier, we consider that it is not reasonable in real life reinsurance to define a cession rate for each policy. For this, either there is one global cession rate then it is a quota share or there are individual cession rates derived from a given retention, and in this case we get surplus reinsurance. Let's write the net result random variable in function of a retention \( R \) and a global cession rate \( \alpha \):

\[ \text{Res}^{\text{net}}(\alpha, R) = \sum_{i=1}^{n} (1 - \alpha) \cdot \min(1, \frac{R}{S_i}) \cdot (p_i - E(S_i)) \]

The differential of the lagrangian for the parameter \( \alpha \) is quite direct but can't be computed for the parameter \( R \) due to the non-linearity of function "minimum". On a numerical example we can already make the following remarks observing the hereunder graph:

\[ \text{Figure 1: Mean net result function of } R \text{ and } \alpha \]

- several combinations of \( (\alpha; R) \) can give the same mean net result
- it is possible to get two combinations of \( (\alpha; R) \) for which we get same mean result but different net variances
- for the same mean net result, net variance is more sensible to decreasing the surplus retention than increasing the quota share cession rate.

As a first conclusion we saw that the resolution gets complicated when trying to implement the theoretical optimum with practicable and existing life reinsurance structures. That drives us in a second part to introduce numerical solving by means of simulation, that will enable us to determine our measures such as Value-at-Risk, minimum and range, S1 requirement and S2 standard formula $SCR$.

4. **THE MODEL AND NUMERICAL OPTIMIZATION**

   The simulation tool *Lifematica* enables us to model from the portfolio detailed line by line, the future cashflows: for the term assurance product, the model generates a random choice between three issues: death, lapse or survival. The cashflows for each state are recorded for each policy in the portfolio: premiums with correct frequency, premium rate by real age, loss amount in case of death, premium loss in case of lapse. We will proceed to 5,000 trials to get stabilized statistical indicators.

   The portfolio we chose is an extract of a client's bigger portfolio. We have 24,500 policies with a total yearly premium of 12 millions euros, with a total sum assured of 2.194 billions. We won't plot here the distribution of sum assured but we can mention that 99% of sum assured are lower than 500K euros, with jumps of 20% 30% and 15% for the sum assured values respectively 25K, 55K and 120K euros. The median sum assured is around 60K euros. Regarding age distribution, the exposure is concentrated in the interval $[30,40]$ years old.

4.1 **Results before reinsurance**

   We can consecutively plot the mean gross result over the years:

   ![Figure 2: Mean gross result, in millions](image)
and the distribution of the gross result on the first year:

![Cumulative distribution function of gross result year 1](image)

*Figure 3: Cumulative distribution function of gross result year 1*

We observe that this distribution is asymmetrical: the distribution of trials under the median is different from the ones over the median. This function has a heavier tail on the adverse scenarios whereas there's a limit in the good results trials. It reflects the asymmetry on the loss distribution function.

On the gross result distribution, we can compute the following noticeable values:

- Average of gross result: 4.4 M
- Standard deviation of gross result: 1.1 M
- Relative standard deviation of gross result: 23.8%
- Minimum of gross result (over 5,000 simulations): -0.2 M
- Maximum of gross result (over 5,000 simulations): 7.5 M

Let's compute the S1 capital requirement: policies have more than 5 years to maturity, then the requirement is equal to 0.3% of the total exposure of 2.194 billion. Hence, S1 capital requirement is 6.6 M.

For the S2 calculation, we compute each $SCR_{Life}$ submodules with corresponding shocks (mortality, lapse up, lapse down, mass, expense & Cat). We can illustrate this calculation with the projection of means premiums over the years for the lapse shocks up & down.
These shocks have only effect on the \( \mathcal{B} \) and not on assets so the variation of \( \mathcal{N} \) is the variation of \( \mathcal{B} \). The \( \mathcal{B} \) before any shock is worth \(-45\) M. We obtain a \( SCR_{\text{life}} \) of \( 21.2 \) M decomposed as follows:

<table>
<thead>
<tr>
<th>Submodule</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCR Life</td>
<td>21 176 466</td>
</tr>
<tr>
<td>SCR Mortality</td>
<td>12 501 244</td>
</tr>
<tr>
<td>SCR Lapse</td>
<td>14 908 377</td>
</tr>
<tr>
<td>SCR Expense</td>
<td>713 754</td>
</tr>
<tr>
<td>SCR CAT</td>
<td>3 153 600</td>
</tr>
</tbody>
</table>

We note that the product is term assurance on death so the disability, longevity and revision submodules are not calculated. Main submodule is the \( SCR_{\text{lapse}} \) followed by the \( SCR_{\text{mortality}} \).

### 4.2 de Finetti mean-variance efficient frontier and structures

In a first analysis, we use the net standard deviation as the risk measure and the net mean result on the first year as performance indicator. Above we determined the expression of the efficient frontier with de Finetti's method. Now we can proceed to numerical application, and getting \( \lambda \) for different values of net mean result \( res_{\text{obj}} \). With \( \lambda \) we can compute each individual cession and thus the standard deviation of net result satisfying the optimality program for each \( res_{\text{obj}} \). Plotting the points of coordinates \( (res_{\text{obj}}, \text{standard deviation}) \) gives us the efficient frontier. Are added the points corresponding to each reinsurance structure we modeled: each point corresponds to a reinsurance combination of a quota share and surplus, with different cession rates, retentions, commission and profit share rates.
Figure 6: Mean variance diagram with optimal and modeled points

The relative positions of points towards the curve reveals that, on a first side, our models are pertinent and on the right side of the curve. On a second side, we look on finding points that can overpass the efficient frontier. There are in top-left quadrant. We remind here that the efficient frontier (reinsurance structures with individual cession rates) was plotted without any commission or profit share clause. Thus, having some points over the curve means that net result is very sensible to these "extra" conditions on each layer.

4.3 Shapes of the efficient frontier with RORAC $S_1$ and $S_2$ against three risk measures

At this point, we are able to quantify the impact (from the gross position) of each reinsurance structure and for all the indicators, regarding performance and risk. We transform a 2-dimensional graph into a 3-dimensional analysis thanks to the following performance rates: $RORAC_{S_1}$ and $RORAC_{S_2}$ as Return on Risk Adjusted Capital, where capital is either the net required capital under Solvency 1 or Solvency 2 ("net" meaning after reinsurance).

$$RORAC_{S_k} = \frac{\text{Mean net result}}{\text{S_k net required capital}}$$

These indicators allow comparison of capital efficiency, as the enable can compute the differential value for one supplementary euro in capital. This is very useful when comparing performance of several lines of business and the allocation of capital in each: reinsurance can help to optimize the allocation of capital against allocation of cost. This means we can find structures for which it is possible:

- to reduce net capital requirement with the same mean net result
- to increase mean net result keeping the same net required capital.
In the following graphs, risk measures are the standard deviation, the TVaR $0.5\%$ and the minimum of distribution. We get in figures 7 to 12 six clouds of points, with in abscissa the risk measure against the performance measure, for each Solvency norm ("Solv.1" and "Solv.2").

Figure 7: Diagram net standard deviation - RORAC$_{S1}$

Figure 8: Diagram net standard deviation - RORAC$_{S2}$

Figure 9: Diagram TVaR $0.5\%$ - RORAC$_{S1}$

Figure 10: Diagram TVaR $0.5\%$ - RORAC$_{S2}$

Figure 11: Diagram Minimum - RORAC$_{S1}$

Figure 12: Diagram Minimum - RORAC$_{S2}$
Each of the six figures shows an explicit outer frontier: this proves the existence of efficient frontier for each combination of risk/reward measure. The most optimal structures are oriented upward to maximize profitability and to the left in the sense of standard deviation, to the right for minimum and $TVaR$. Of course, all the structures modeled here are in reasonable domain of definition: 100% quota-shares or quota-share with advance commission or other alternative structures are not plotted here and are discussed later in this article.

We notice that form and orientation are specifically defined for a couple (risk,reward). We then ask the following question: does the hierarchy in the structures changes if we choose another reward measure or another risk measure, or both?

We use the points 1 and 2 on the six diagrams, and start from figure 7.

- from norm S2 to norm S1 (fig 7): for point 2, net standard deviation remains the same but profitability gets further from efficient frontier. Same effect for point 1. Relative position of both points remains the same.

- from standard deviation to $TVaR_{0.5\%}$ (fig 10): point 2 is on the edge of the cloud, but not on the efficient part: there are points with better profitability (>16%) for the same TVaR (1.5M). Point 1 is close to efficient frontier, and in this case the order is inversed: point 1 is more optimal than point 2.

- from standard deviation to Minimum (fig 12): point 1 keeps its position but point 2 gets further from the frontier: it is no longer efficient.

- from norm S2 and standard deviation to norm S1 and Minimum (fig 11): same comments except that point 1 is neither optimal.

- from norm S2 and standard deviation to norm S1 and $TVaR_{0.5\%}$ (fig 9): point 1 is far from efficient frontier and for the same profitability, other points maximize TVaR. Point 2 remains on the frontier and we can say it is one of the last points to remain efficient as there is a breaking point in the curve. No matter this, it is more efficient in the sense of norm S1 profitability and $TVaR_{0.5\%}$ than point 1.
Enumerating the transitions and all effects on relative position of points to the efficient frontier highlights the difficulty to find a global efficiency, and further more a global optimum. Thus, such an observation encourages to conduct solving of $n$ dimensions problems. And then, the solution introducing the stochastic dominance (order 1 and 2) seems pertinent: it takes every point of the cumulative distribution function into account. Still, it is a comparison method, and an exhaustive pairs comparison can become heavy and fastidious. Acceleration algorithms for the portfolio selection exist but it is not the subject here.

4.4 Specificity of reinsurance structures - the duration

The duration of the reinsurance is a very important parameter of the treaty. According to Solvency 2 boundaries contracts definition, the reinsurance must be taken for the contractual years. Hence a 1-year treaty has S2 effect only on the first year of the whole best estimate projections. Thus, the calibration of treaties must be done on form, levels but also on duration. The stochastic optimization problem is now 3-dimensional. These considerations encourage us to introduce new measure taking account of the lifetime of the reinsurance: the net $VIF$ (Value In Force, i.e. discounted expected value of future technical results) replaces the mean net result on first year.

The two main indicators where duration has an impact are on $VIF$ and capital. For this, we are going to study the future cashflows net of reinsurance, as long as the reinsurance is in place (duration of the treaty). In Solvency 1, the capital requirement is calculated on total exposure, the duration of treaty has then no impact. We can plot the saving in capital required against the cost of reinsurance (cost of reinsurance is the ceded result, after recoveries, commissions, profit share, ...) and the following diagram will then serve as a comparison.
We note that we also compared with the profit sharing parameter, to visualize how it affects the counterparty risk. In S1, we observe only a movement to the left as more result is returned to the cedant on the reinsurance technical account. Besides the effect on the economy of capital is dependent of the form of the reinsurance - quota share or surplus.

Let’s now see the effect of duration on the economy under Solvency 2. With the same structures and the same profit sharing parameters, we plot the effects of the quota-share and surplus on S2 capital economy against the cession of VIF.

Figure 13: Economy in Solvency 1 capital requirement against cost

Figure 14: Economy in Solvency 2 capital requirement against cost, with 1, 3 and 30-years durations
Increasing the duration enables to cede more capital requirement to reinsurance: 1 year allows to cede at maximum 10% where a reinsurance per generation (> 30 years) allows ceding a maximum of more than 65%. We also notice that the translation to the left is still due to the profit sharing clause that returns some result to the cedant. Besides profit sharing is subject to default risk, hence it charges the counterparty risk submodule when the profit sharing rate increases: that's why left curves are slightly lower than the one on the right (without profit share). Below we show an example of the evolution of the $\text{SCR}_{\text{underwriting}}$ and the $\text{SCR}_{\text{counterparty}}$ in function of the duration of the reinsurance treaty.

![Figure 15: SCRs in function of the duration of a quota share treaty](image)

To sum up our analysis on proportional structures, we can assess the S1 capital economy is mainly driven by the form of the reinsurance whereas the S2 one is mainly driven by the duration of the reinsurance. There is also an interest in swapping underwriting risk with counterparty risk that is less penalizing for the cedant in terms of S2 requirement.

### 4.5 Non-proportional reinsurance and alternatives

Now that we plotted efficient frontiers for proportional structures on a life portfolio, we are able to characterize the relative positions of non-proportional structures towards these frontiers. Indeed, we started with proportional as price is given by the parameters of the structures, and our efficient frontiers include all variations of commercial and duration parameters, hence they are independent from price considerations. The latter is the only additional parameter for non-proportional structures, and thus our graphs enable to tell us in which price range of excess structures (non-prop.) the proportional ones are remaining optimal or overpassed by these excess structures. Let's discuss below their effects on our
measures: profit, deviation and extreme risk, capital requirements under Solvency 1 and Solvency 2. Among non-proportional reinsurance we will distinguish aggregate covers and per risk covers.

**Excess of loss per risk**

On the life reinsurance market, *per risk* means *per head* or *per policy*. Let's consider independently two reinsurance structures on the same portfolio, on one side an excess of loss per head and on the other side a surplus. We assume that retention and limit are set on both structures so that they deliver the same amount of recoveries. Duration, profit share and commission for both structures, and above all price of the excess of loss per head, have to be set. The existence of the efficient frontiers enable us to tell which conditions make one structure or the other better. A simple exercise is to set both durations to one year, commission and profit share rates to zero. Then the price of the non proportional structure is the only parameter left (ceded premiums for the surplus are calculated with the retention level). We can easily plot on the graphs with efficient frontiers the lines corresponding to the excess of loss function of its price. These lines should be straight and vertical for standard deviation in S1: price does not impact standard deviation nor S1 capital requirement, and only mean result is linear to it. To account for excess per risk effect in S2 balance sheet, the duration of the treaties has to be expended - more than the usual 1 to 3 years. But this drives us back to the discussion on prices and risk analysis from reinsurers, as duration adds more uncertainty on the losses.

**Excess of loss aggregates: Cat or Stop-loss**

A Cat excess of loss is attached to the occurrence of an event defined by the treaty: the loss is concentrated in time and originated from the same cause. Such a structure has an effect in case of catastrophic event on the technical result of course. Besides in terms of Solvency, it can be used to cover the Cat submodule of the $SCR_{Life}$, but care has to be taken in the definition of the event implying mortality increase. Still, it seems more appropriate to study an aggregate structure to cover a mortality increase not necessarily linked to a catastrophic event, and the excess is applied on the cumulated values of the losses that overpasses a certain amount (the retention): the reinsurance takes to its charge the overload as long as it does go over the limit of the excess of loss treaty.

A stop-loss (i.e. no-event linked aggregate excess of loss) has effect on mean result
through its price and protects extreme scenarios (minimum and TVaR); hence effect on RORAC under Solvency 1 is quite direct and depends exclusively on its price and level of retention. Under Solvency 2, same comments as for the per risk excess can be made, as far as duration of the reinsurance is the main factor reducing capital requirement. The level of retention is of course very important as well, as it determines if the stop-loss covers the S2 shocks (mortality shock, disability, ...) i.e. reduces best estimates liabilities.

Thus, it illustrates that having efficient frontiers for one type of reinsurance allows comparability for all others for which an essential parameter is exogenous: the price.

Alternative structures

We mentioned above that if the parameters of reinsurance are fixed to the limits (example of 100% quota share), the structure can no longer remain in the traditional domain - financing reinsurance, monetization or margin solutions. In those cases, structuring faces regulation limits, and the latter almost represents a fourth dimension to optimization study. Indeed, transferring 100% of a portfolio asks if the transfer would not rather be a complete sale of the portfolio, and in this case the transfer is no longer made under reinsurance conditions, even if the accounting is defined in a reinsurance treaty. The structuring steps just as the pricing are then very important as well as data analysis from all parties to consider if there is a risk transfer or not.

4.6 Parameter vs. perimeter

We studied the construction of efficient frontiers on a single and run-off portfolio. A more global approach can consist in using reinsurance perimeter as a decisional variable. This drives to consider the reinsurance as a balance sheet tool with a wider vision on portfolios of the same cedant, helping:

- the reinsurer to have a better knowledge of the cedant
- the reinsurer to have more competitive pricing thanks this knowledge
- to transfer natural hedging (e.g. joint cession of longevity and mortality guarantees)
- to set compensation mechanisms between ceded portfolios or risks

But such structuring has to be made with a lot of care as the enlargement of perimeter includes each risk specificity and naturally complicates the analysis, especially
where assets are concerned (like in savings or annuities) or common profit share clauses between ceded products. In any case, as mentioned in the motivation of this paper, the increase in the perimeter implicates more information and data needs.

5. A LOOK INTO THE EQUATIONS OF EFFICIENT FRONTIERS WITH NORMAL APPROXIMATION

In previous section, we simulated the death and lapses randomly. We could expect with this method to get, before any reinsurance, a normal distribution of the result, as limit of a sum of Bernoulli distributions.

5.1 Normality test

In order to judge if the modeled random variable of the result is in line with a normal distribution, we compute each percentiles for both distributions, before any reinsurance issue, that have same mean and variance (parameters of the normal distribution). We plot below the QQ - plot for these two distributions:

![Figure 16: QQ-plot of normal distribution against modeled distribution](image)

It reveals that our modeled distribution is asymmetrical: although central percentiles are very similar, we get more spread on the tails. Indeed, our result is limited in positive values whereas it is worse than normal distribution on the negative result values.

The first reason that comes to mind is that the number of trials is not enough: we
conduct the same simulation but we 25 000 trials, but we get the same observations. We can also question the modeling method to record cashflows: to stick to an accounting reality, we randomly chose a date in the year for the lapse of the death; if the event occurs before the payment date of the premium, the premium is not recorded (the policy is out of the portfolio before payment). Thus, this worsens the losses as more premium is not earned.

Finally the most influencing cause is likely to be the disparity in distributions of age and sum insured of the portfolio. Both central limit theorem and convergence of binomial random variables require independent and identically distributed random variables. In our model, the loss \( X_i = K_i \cdot 1_{\text{Usk}_i} \) for each policy are independent, but the ranges of sum insured and age make that \( X_i \) are not identically distributed.

Still, on larger portfolios, we could invoke convergence of independent Bernoulli random variables that are bounded. This condition is respected in case of term assurance on death as payments for all policies are bounded by the contractual sum insured. Thus, this convinces us that convergence to normal distribution is possible but is slower than with i.i.d condition.

5.2 Equations and propositions

In this section, we try to determine the formal relations between statistics of gross and net result, directly in function of reinsurance parameters. The goal of this work is to be able to characterize transformations of net distributions so that we can write the optimization programs without the reinsurance form constraint. Two questions naturally arise: does the result random variable keep the same distribution, and if it is the case, how are transformed the parameters?

In the following, we suppose the gross result random variable to be normally distributed: \( \text{Res}^{\text{gross}} \sim \mathcal{N}(\mu_{\text{gross}}, \nu_{\text{gross}}) \). It is important to note that normal distribution assumption makes us find only two transformation to characterize the net normal distribution (if the net remains normally distributed, we will see that in the following): transformation on mean \( T_1 \) and variance \( T_2 \).

Quota share

In the case of quota share with cession rate \( \alpha \), we have that the result is normally distributed with

\[
\tau_1^{\text{QS}_a} = \tau_2^{\text{QS}_a} = (1 - \alpha) \cdot \text{Id}
\]

where \( \text{Id} \) is the transformation Identity. The same effect is observed on all percentiles.
Surplus

The statistics net of reinsurance are not linear in function of the retention level. For this reason, we transform our data with the function $g(x) = \ln(\mu_{\text{gross}} - x)$. By this transform we observe a linear relation between the statistic (here the mean $\mu$) and the retention level.

Using a solver to minimize residuals on the two lines, we obtain the following equation for the mean ($R$ is the retention level):

$$\ln(\mu_{\text{gross}} - \mu(R)) = f_1(R) + f_2(R)$$

that gives

$$\mu(R) = \mu_{\text{gross}} - e^{f_1(R) + f_2(R)}$$

where

$$f_1(R) = (a_1R + b_1) \cdot 1_{R \leq 350000}$$
$$f_2(R) = (a_2R + b_2) \cdot 1_{R > 350000}$$

We proceed with the same method on the standard deviation, obtaining the same expressions, that enable us to plot estimations of mean and standard deviation:
Besides, the Bernoulli random variables net of surplus remain Bernoulli random variables (the sum assured is reduced of the individual cession rate), and then as the sum of bounded independent Bernoulli random variables, the result is normally distributed.

\[ Res^{net} \sim N(T_1^{SPR}(\mu_{gross}); T_2^{SPR}(v_{gross})) \]

where transforms are expressed as follows:

\[ T_1^{SPR} = 1d - e^{A_1 \cdot R + B_1} \]
\[ T_2^{SPR} = 1d - e^{A_2 \cdot R + B_2} \]

We supposed that the surplus limit is over the maximum sum insured of the portfolio and fixed for all retentions. Similar results are obtained if the limit is applied for a retention level as it only changes the bounds in the range of sum assured.

**Excess of loss**

If we consider a per-head (=per risk) non-proportional reinsurance, liabilities cashflows are the same as for surplus, only the reinsurance premium changes. It is no longer calculated of original premiums (originally earned by the cedant) but is priced on a risk-based approach by reinsurers. The technical result is the sum of \( n \) truncated Bernoulli distributions, so for the same reasons, \( Res^{net} \sim N(T_1^{SPR}(\mu_{gross}); T_2^{SPR}(v_{gross})) \) with \( T_1^{XSR} \) and \( T_2^{XSR} \). The difference with the surplus lies in the calibration of the parameters as the reinsurance premium deforms the net result curve.

Now if we look at aggregate structures, the reinsurance is applied on the cumulated amount of loss. If the gross result is normally distributed, so are the losses. Net of aggregate on losses reinsurance, the result is a truncated to the left normal distribution. Indeed, if \( S \) represents the losses, \( P \) premiums and \( RetAgg \) the retention for an aggregate cover,

\[ Res^{net} = (P^{gross} - P^{reins}) - \min(RetAgg; S^{gross}) \]

where \( S^{gross} \) follows normal distribution.
Effect of commissions and profit sharing

The commission cashflow is based on the premiums cashflow and then its variability depends on the premiums. If we suppose fixed premiums in the projection to simplify expressions, the distribution is not altered by commissions, except on the level of the constants, but the expressions of transforms remain identical.

As regards the profit sharing, we can expect that it will be different. Indeed, the profit share calculation is based on the ceded result which is a random variable, hence there is a return of variability. We are going to write the equations for a quota share with profit sharing, writing $b$ the amount of the reinsurance fees amount on profit share,

$$Res^{net} = (1 - \alpha) \cdot Res^{gross} + t_{ps} \cdot (\alpha \cdot Res^{gross} - b)$$

With normality assumption, the expression of the mean net result is:

$$E(Res^{net}) = \int_{-\infty}^{+\infty} [(1 - \alpha)x + \alpha t_{ps}(x - \frac{b}{\alpha})^+] \cdot f_{Res^{gross}}(x)dx$$

$$= (1 - \alpha) \cdot \int_{-\infty}^{+\infty} xf_{Res^{gross}}(x)dx + \alpha t_{ps} \cdot \int_{\frac{b}{\alpha}}^{+\infty} (x - \frac{b}{\alpha})f_{Res^{gross}}(x)dx$$

$$= (1 - \alpha) \cdot \mu_{gross} + \alpha t_{ps} \cdot E(X|X > \frac{b}{\alpha}) - \alpha t_{ps} \cdot P(X > \frac{b}{\alpha})$$

The mean transformation is:

$$T_{1}^{QuotaPS} = Id - \alpha \cdot (Id - t_{ps} \cdot E(X|X > \frac{b}{\alpha}) - P(X > \frac{b}{\alpha}))$$

Similar expression for $T_{2}$ on variance can be found.

Propositions

As a conclusion to this section, we showed under the normal-distributed gross result assumption, that:

- result net of aggregate excess of loss has truncated normal distribution
- result net of quota share, surplus and excess per risk, when there is no profit sharing, has normal distribution
- and the distribution is determined for measure $\rho$ (mean or variance) by transform

$$T_{\rho}: R = \rho_{gross} - \frac{at_{ps}}{\sqrt{\tau_{ps}}}$$

- with profit sharing, net result is the sum of normal and truncated normal distribution

Using the normal approximation, like the simulation, enabled us to pass over the non linearity of the minimum function which defines each cashflows net of quota share or
surplus/excess of loss. Further study on the truncated distributions and the characterization of the break point in surplus expression for a dataset will be conducted, as well as the formalization of the new optimization programs with the propositions above. The normal assumption will for example facilitate the expression of VaR on gross and net results.

6. CONCLUSION

Our method consisted in a risk-return approach by analogy and comparison to Markowitz and de Finetti analyses. We considered different reinsurance structures for several risk measures and two regulatory environments.

The modeling with actuarial software Lifemetrica™ (Aon Benfield (2012)) enabled us to first plot efficient frontiers for mean-variance for a real life portfolio, but we also managed to go further by plotting an efficient frontier for any combination of performance and risk measure. Lastly, the normal assumption made possible to characterize transformation on the inputs of the normal gross distribution - this assumption allows now to proceed to optimization for group life insurance, when line to line data is not known and when the studied portfolio is large enough to conduct optimality study. Results on continuous optimization for non-linear functions can complete this analysis.

Our study aimed at introducing a method that allows reinsurance buyers to compare reinsurance structures on any criterion, and for which optimal values can be found depending foremost on the complexity of the optimization programs. On top of this, we mentioned that duration of reinsurance treaty is an implicit parameter in the projections and represents a essential dimension for optimization, and that combining several risks enables to modify perimeter to reach global optimality.

Thus, the traditional risk management vision of reinsurance is being challenged by its quantifiable ability to arbitrate capital needs, from one single portfolio to different lines of business, becoming this way a key strategic tool.

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